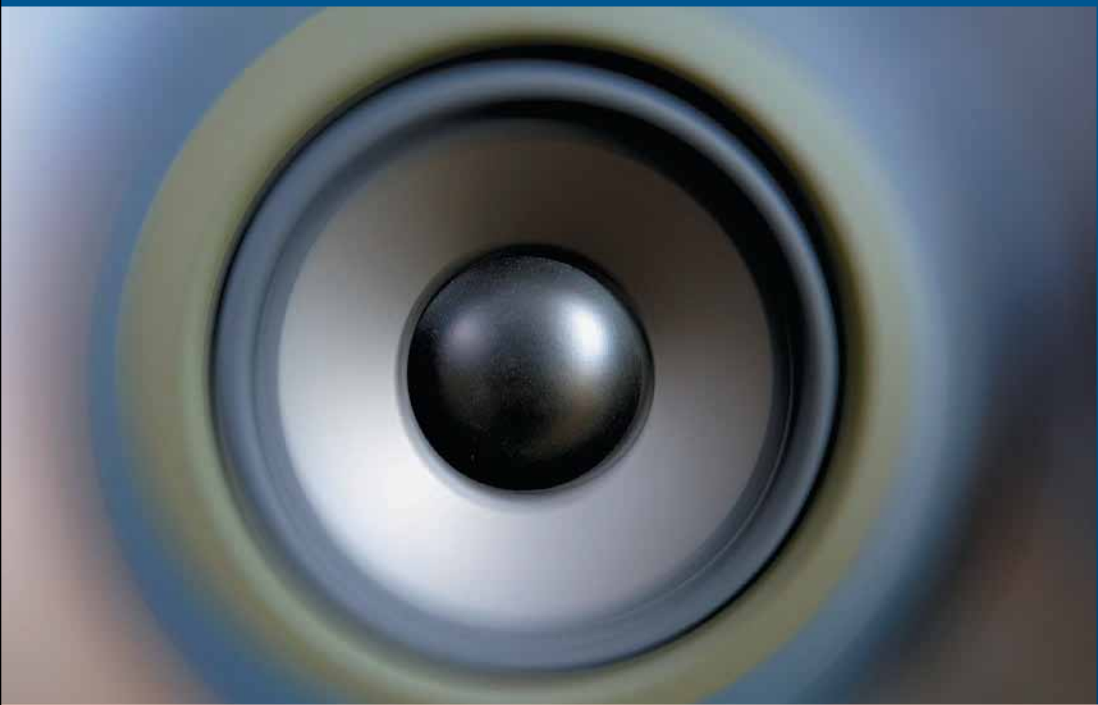
14 Oscillations

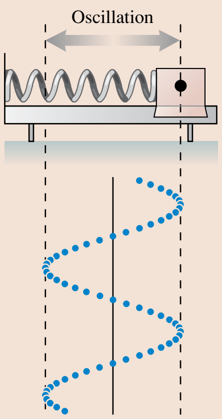


This loudspeaker cone generates sound waves by oscillating back and forth at audio frequencies.

▶ **Looking Ahead** The goal of Chapter 14 is to understand systems that oscillate with simple harmonic motion.

Simple Harmonic Motion

The most basic oscillation, with sinusoidal motion, is called **simple harmonic motion.**



The oscillating cart is an example of simple harmonic motion. You'll learn how to use the mass and the spring constant to determine the frequency of oscillation.

In this chapter you will learn to:

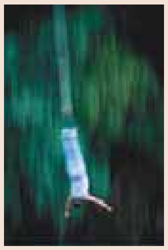
- Represent simple harmonic motion both graphically and mathematically.
- Understand the dynamics of oscillating systems.
- Recognize the similarities among many types of oscillating systems.

Simple harmonic motion has a very close connection to uniform circular motion. You'll learn that an edge-on view of uniform circular motion is none other than simple harmonic motion.

◀ **Looking Back**
Section 4.5 Uniform circular motion

Springs

Simple harmonic motion occurs when there is a **linear restoring force.** The simplest example is a mass on a spring. You will learn how to determine the period of oscillation.



The "bounce" at the bottom of a bungee jump is an exhilarating example of a mass oscillating on a spring.

◀ **Looking Back**
Section 10.4 Restoring forces

Pendulums

A mass swinging at the end of a string or rod is a **pendulum.** Its motion is another example of simple harmonic motion.

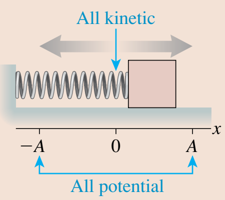
The period of a pendulum is determined by the length of the string; neither the mass nor the amplitude matters. Consequently, the pendulum was the basis of time keeping for many centuries.

Illustrated Chapter Previews give an overview of the upcoming ideas for each chapter, setting them in context, explaining their utility, and tying them to existing knowledge (through Looking Back references). These previews build on the cognitive psychology concept of an "advance organizer."

Energy of Oscillations

If there is no friction or other dissipation, then the mechanical energy of an oscillator is conserved. Conservation of energy will be an important tool.

The system oscillates between all kinetic energy and all potential energy

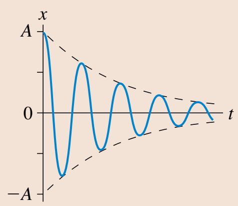


◀ **Looking Back**
Section 10.5 Elastic potential energy
Section 10.6 Energy diagrams

Damping and Resonance

If there's drag or other dissipation, then the oscillation "runs down." This is called a **damped oscillation.**

The amplitude of a damped oscillation undergoes *exponential decay.*



Oscillations can increase in amplitude, sometimes dramatically, when driven at their natural oscillation frequency. This is called **resonance.**

FIGURE 14.1 Examples of position-versus-time graphs for oscillating systems.

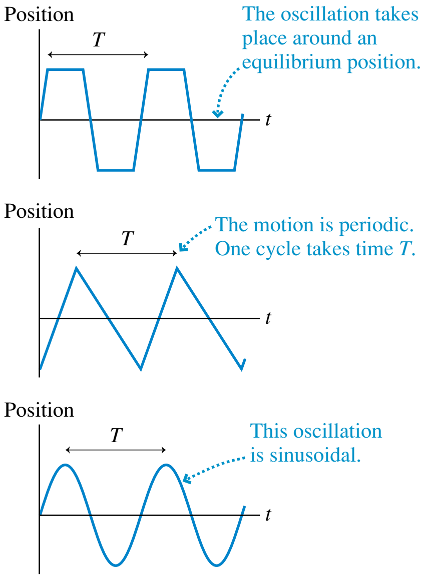


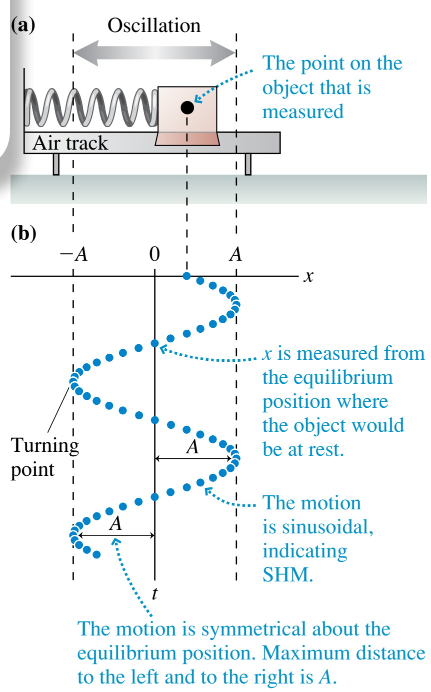
TABLE 14.1 Units of frequency

Frequency	Period
10^3 Hz = 1 kilohertz = 1 kHz	1 ms
10^6 Hz = 1 megahertz = 1 MHz	1 μs
10^9 Hz = 1 gigahertz = 1 GHz	1 ns

Figures are carefully **streamlined in detail and color** so students focus on the physics—for instance, the object of interest in mechanics.

Explicit instruction as **annotations directly on figures** helps students to interpret figures and graphs.

FIGURE 14.2 A prototype simple-harmonic-motion experiment.



14.1 Simple Harmonic Motion

Objects or systems of objects that undergo **oscillatory motion**—a repetitive motion back and forth around an equilibrium position—are called **oscillators.** FIGURE 14.1 shows position-versus-time graphs for three different oscillating systems. Although the shapes of the graphs are different, all these oscillators have two things in common:

1. The oscillation takes place about an equilibrium position, and
2. The motion is *periodic*, repeating at regular intervals of time.

The time to complete one full cycle, or one oscillation, is called the **period** of the motion. Period is represented by the symbol T.

A closely related piece of information is the number of cycles, or oscillations, completed per second. If the period is $\frac{1}{10}$ s, then the oscillator can complete 10 cycles in one second. Conversely, an oscillation period of 10 s allows only $\frac{1}{10}$ of a cycle to be completed per second. In general, T seconds per cycle implies that $1/T$ cycles will be completed each second. The number of cycles per second is called the **frequency** f of the oscillation. The relationship between frequency and period is

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f} \tag{14.1}$$

The units of frequency are **hertz,** abbreviated Hz, named in honor of the German physicist Heinrich Hertz, who produced the first artificially generated radio waves in 1887. By definition,

$$1 \text{ Hz} \equiv 1 \text{ cycle per second} = 1 \text{ s}^{-1}$$

We will frequently deal with very rapid oscillations and make use of the units shown in Table 14.1.

NOTE ▶ Uppercase and lowercase letters *are* important. 1 MHz is 1 megahertz = 10^6 Hz, but 1 mHz is 1 millihertz = 10^{-3} Hz! ◀

EXAMPLE 14.1 Frequency and period of a loudspeaker cone

What is the oscillation period of a loudspeaker cone that vibrates back and forth 5000 times per second?

SOLVE The oscillation frequency is $f = 5000$ cycles/s $= 5000$ Hz $= 5.0$ kHz. The period is the inverse of the frequency; hence

$$T = \frac{1}{f} = \frac{1}{5000 \text{ Hz}} = 2.0 \times 10^{-4} \text{ s} = 200\ \mu\text{s}$$

A system can oscillate in many ways, but we will be especially interested in the smooth *sinusoidal* oscillation (i.e., like a sine or cosine) of the third graph in Figure 14.1. This sinusoidal oscillation, the most basic of all oscillatory motions, is called **simple harmonic motion,** often abbreviated SHM. Let's look at a graphical description before we dive into the mathematics of simple harmonic motion.

FIGURE 14.2a shows an air-track glider attached to a spring. If the glider is pulled out a few centimeters and released, it will oscillate back and forth on the nearly frictionless air track. FIGURE 14.2b shows actual results from an experiment in which a computer was used to measure the glider's position 20 times every second. This is a position-versus-time graph that has been rotated 90° from its usual orientation in order for the x-axis to match the motion of the glider.

The object's maximum displacement from equilibrium is called the **amplitude** A of the motion. The object's position oscillates between $x = -A$ and $x = +A$. When using a graph, notice that the amplitude is the distance from the *axis* to the maximum, *not* the distance from the minimum to the maximum.

FIGURE 14.3a shows the data with the graph axes in their "normal" positions. You can see that the amplitude in this experiment was $A = 0.17$ m, or 17 cm. You can also measure the period to be $T = 1.60$ s. Thus the oscillation frequency was $f = 1/T = 0.625$ Hz.

FIGURE 14.3b is a velocity-versus-time graph that the computer produced by using $\Delta x/\Delta t$ to find the slope of the position graph at each point. The velocity graph is also sinusoidal, oscillating between $-v_{max}$ (maximum speed to the left) and $+v_{max}$ (maximum speed to the right). As the figure shows,

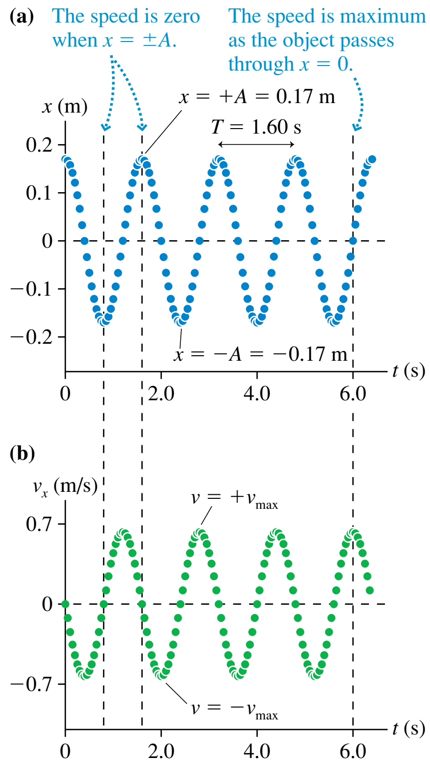
- The instantaneous velocity is zero at the points where $x = \pm A$. These are the *turning points* in the motion.
- The maximum speed v_{max} is reached as the object passes through the equilibrium position at $x = 0$ m. The *velocity* is positive as the object moves to the right but *negative* as it moves to the left.

We can ask three important questions about this oscillating system:

1. How is the maximum speed v_{max} related to the amplitude A?
2. How are the period and frequency related to the object's mass m, the spring constant k, and the amplitude A?
3. Is the sinusoidal oscillation a consequence of Newton's laws?

A mass oscillating on a spring is the prototype of simple harmonic motion. Our analysis, in which we answer these questions, will be of a spring-mass system. Even so, most of what we learn will be applicable to other types of SHM.

FIGURE 14.3 Position and velocity graphs of the experimental data.



Kinematics of Simple Harmonic Motion

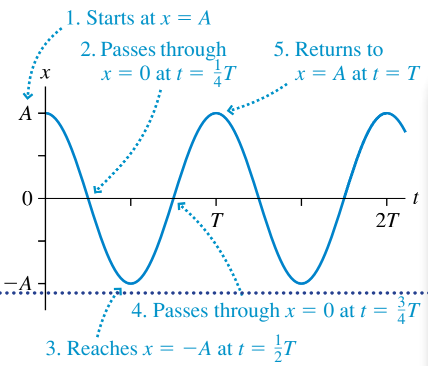
FIGURE 14.4 redraws the position-versus-time graph of Figure 14.3a as a smooth curve. Although these are empirical data (we don't yet have any "theory" of oscillation) the graph for this particular motion is clearly a cosine function. The object's position is

$$x(t) = A\cos\left(\frac{2\pi t}{T}\right) \tag{14.2}$$

where the notation $x(t)$ indicates that the position x is a *function* of time t. Because $\cos(2\pi) = \cos(0)$, it's easy to see that the position at time $t = T$ is the same as the position at $t = 0$. In other words, this is a cosine function with period T. Be sure to convince yourself that this function agrees with the five special points shown in Figure 14.4.

NOTE ▶ The argument of the cosine function is in *radians*. That will be true throughout this chapter. It's especially important to remember to set your calculator to radian mode before working oscillation problems. Leaving it in degree mode will lead to errors. ◀

FIGURE 14.4 The position-versus-time graph for simple harmonic motion.



NOTE paragraphs throughout guide students away from known preconceptions and around common sticking points and highlight many math- and vocabulary-related issues that have been proven to cause difficulties.

We can write Equation 14.2 in two alternative forms. Because the oscillation frequency is $f = 1/T$, we can write

$$x(t) = A\cos(2\pi f t) \tag{14.3}$$

Recall from Chapter 4 that a particle in circular motion has an *angular velocity* ω that is related to the period by $\omega = 2\pi/T$, where ω is in rad/s. Now that we've defined the frequency f, you can see that ω and f are related by

$$\omega \text{ (in rad/s)} = \frac{2\pi}{T} = 2\pi f \text{ (in Hz)} \tag{14.4}$$

In this context, ω is called the **angular frequency.** The position can be written in terms of ω as

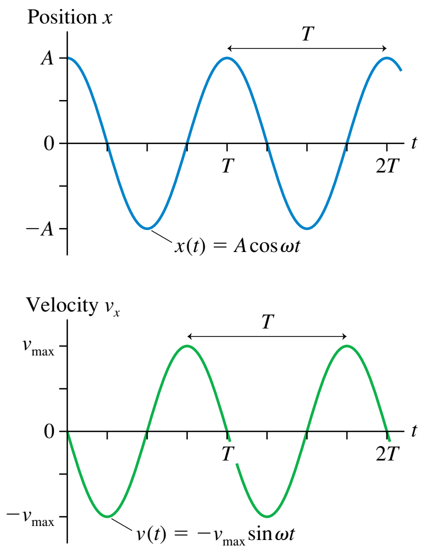
$$x(t) = A\cos\omega t \tag{14.5}$$

Equations 14.2, 14.3, and 14.5 are equivalent ways to write the position of an object moving in simple harmonic motion.

TABLE 14.2 Derivatives of sine and cosine functions

$\frac{d}{dt}\big(a\sin(bt+c)\big) = +ab\cos(bt+c)$
$\frac{d}{dt}\big(a\cos(bt+c)\big) = -ab\sin(bt+c)$

FIGURE 14.5 Position and velocity graphs for simple harmonic motion.



Just as the position graph was clearly a cosine function, the velocity graph shown in **FIGURE 14.5** is clearly an "upside-down" sine function with the same period T. The velocity v_x, which is a function of time, can be written

$$v_x(t) = -v_{max}\sin\left(\frac{2\pi t}{T}\right) = -v_{max}\sin(2\pi f t) = -v_{max}\sin\omega t \qquad (14.6)$$

NOTE ▶ v_{max} is the maximum *speed* and thus is a *positive* number. ◀

We deduced Equation 14.6 from the experimental results, but we could equally well find it from the position function of Equation 14.2. After all, velocity is the time derivative of position. Table 14.2 on the previous page reminds you of the derivatives of the sine and cosine functions. Using the derivative of the position function, we find

$$v_x(t) = \frac{dx}{dt} = -\frac{2\pi A}{T}\sin\left(\frac{2\pi t}{T}\right) = -2\pi f A\sin(2\pi f t) = -\omega A\sin\omega t \quad (14.7)$$

Comparing Equation 14.7, the mathematical definition of velocity, to Equation 14.6, the empirical description, we see that the maximum speed of an oscillation is

$$v_{max} = \frac{2\pi A}{T} = 2\pi f A = \omega A \qquad (14.8)$$

Equation 14.8 answers the first question we posed above, which was how the maximum speed v_{max} is related to the amplitude A. Not surprisingly, the object has a greater maximum speed if you stretch the spring farther and give the oscillation a larger amplitude.

Worked examples follow a consistent problem-solving strategy and include careful explanations of the underlying, and often unstated, reasoning.

EXAMPLE 14.2 A system in simple harmonic motion

An air-track glider is attached to a spring, pulled 20.0 cm to the right, and released at $t = 0$ s. It makes 15 oscillations in 10.0 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What are the position and velocity at $t = 0.800$ s?

MODEL An object oscillating on a spring is in SHM.

SOLVE a. The oscillation frequency is

$$f = \frac{15 \text{ oscillations}}{10.0 \text{ s}} = 1.50 \text{ oscillations/s} = 1.50 \text{ Hz}$$

Thus the period is $T = 1/f = 0.667$ s.

b. The oscillation amplitude is $A = 0.200$ m. Thus

$$v_{max} = \frac{2\pi A}{T} = \frac{2\pi(0.200 \text{ m})}{0.667 \text{ s}} = 1.88 \text{ m/s}$$

c. The object starts at $x = +A$ at $t = 0$ s. This is exactly the oscillation described by Equations 14.2 and 14.6. The position at $t = 0.800$ s is

$$x = A\cos\left(\frac{2\pi t}{T}\right) = (0.200 \text{ m})\cos\left(\frac{2\pi(0.800 \text{ s})}{0.667 \text{ s}}\right)$$
$$= (0.200 \text{ m})\cos(7.54 \text{ rad}) = 0.0625 \text{ m} = 6.25 \text{ cm}$$

The velocity at this instant of time is

$$v_x = -v_{max}\sin\left(\frac{2\pi t}{T}\right) = -(1.88 \text{ m/s})\sin\left(\frac{2\pi(0.800 \text{ s})}{0.667 \text{ s}}\right)$$
$$= -(1.88 \text{ m/s})\sin(7.54 \text{ rad}) = -1.79 \text{ m/s} = -179 \text{ cm/s}$$

At $t = 0.800$ s, which is slightly more than one period, the object is 6.25 cm to the right of equilibrium and moving to the *left* at 179 cm/s. Notice the use of radians in the calculations.

EXAMPLE 14.3 Finding the time

A mass oscillating in simple harmonic motion starts at $x = A$ and has period T. At what time, as a fraction of T, does the object first pass through $x = \frac{1}{2}A$?

SOLVE Figure 14.4 showed that the object passes through the equilibrium position $x = 0$ at $t = \frac{1}{4}T$. This is one-quarter of the total distance in one-quarter of a period. You might expect it to take $\frac{1}{8}T$ to reach $\frac{1}{2}A$, but this is not the case because the SHM graph is not linear between $x = A$ and $x = 0$. We need to use $x(t) = A\cos(2\pi t/T)$. First, we write the equation with $x = \frac{1}{2}A$:

$$x = \frac{A}{2} = A\cos\left(\frac{2\pi t}{T}\right)$$

Then we solve for the time at which this position is reached:

$$t = \frac{T}{2\pi}\cos^{-1}\left(\frac{1}{2}\right) = \frac{T}{2\pi}\frac{\pi}{3} = \frac{1}{6}T$$

ASSESS The motion is slow at the beginning and then speeds up, so it takes longer to move from $x = A$ to $x = \frac{1}{2}A$ than it does to move from $x = \frac{1}{2}A$ to $x = 0$. Notice that the answer is independent of the amplitude A.

STOP TO THINK 14.1 An object moves with simple harmonic motion. If the amplitude and the period are both doubled, the object's maximum speed is

a. Quadrupled. b. Doubled. c. Unchanged.
d. Halved. e. Quartered.

14.2 Simple Harmonic Motion and Circular Motion

The graphs of Figure 14.5 and the position function $x(t) = A\cos\omega t$ are for an oscillation in which the object just happened to be at $x_0 = A$ at $t = 0$. But you will recall that $t = 0$ is an arbitrary choice, the instant of time when you or someone else starts a stopwatch. What if you had started the stopwatch when the object was at $x_0 = -A$, or when the object was somewhere in the middle of an oscillation? In other words, what if the oscillator had different *initial conditions*. The position graph would still show an oscillation, but neither Figure 14.5 nor $x(t) = A\cos\omega t$ would describe the motion correctly.

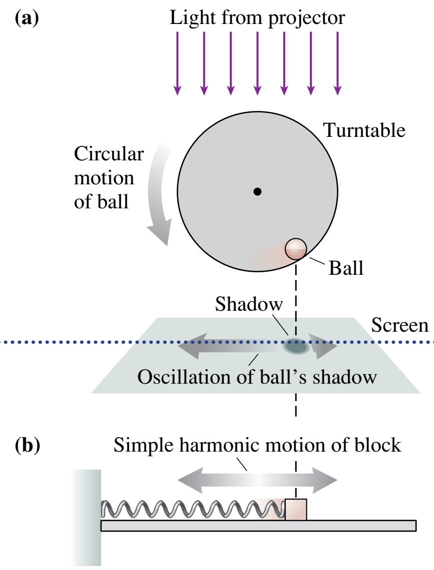
To learn how to describe the oscillation for other initial conditions it will help to turn to a topic you studied in Chapter 4—circular motion. There's a very close connection between simple harmonic motion and circular motion.

Imagine you have a turntable with a small ball glued to the edge. FIGURE 14.6a shows how to make a "shadow movie" of the ball by projecting a light past the ball and onto a screen. The ball's shadow oscillates back and forth as the turntable rotates. This is certainly periodic motion, with the same period as the turntable, but is it simple harmonic motion?

New concepts are introduced through observations about the real world and theories, grounded by making sense of observations. This **inductive approach** illustrates how science operates, and has been shown to improve student learning by reconciling new ideas with what they already know.

To find out, you could place a real object on a real spring directly below the shadow, as shown in FIGURE 14.6b. If you did so, and if you adjusted the turntable to have the same period as the spring, you would find that the shadow's motion exactly matches the simple harmonic motion of the object on the spring. **Uniform circular motion projected onto one dimension is simple harmonic motion.**

FIGURE 14.6 A projection of the circular motion of a rotating ball matches the simple harmonic motion of an object on a spring.



To understand this, consider the particle in FIGURE 14.7. It is in uniform circular motion, moving *counterclockwise* in a circle with radius A. As in Chapter 4, we can locate the particle by the angle ϕ measured ccw from the x-axis. Projecting the ball's shadow onto a screen in Figure 14.6 is equivalent to observing just the x-component of the particle's motion. Figure 14.7 shows that the x-component, when the particle is at angle ϕ, is

$$x = A\cos\phi \tag{14.9}$$

Recall that the particle's *angular velocity,* in rad/s, is

$$\omega = \frac{d\phi}{dt} \tag{14.10}$$

This is the rate at which the angle ϕ is increasing. If the particle starts from $\phi_0 = 0$ at $t = 0$, its angle at a later time t is simply

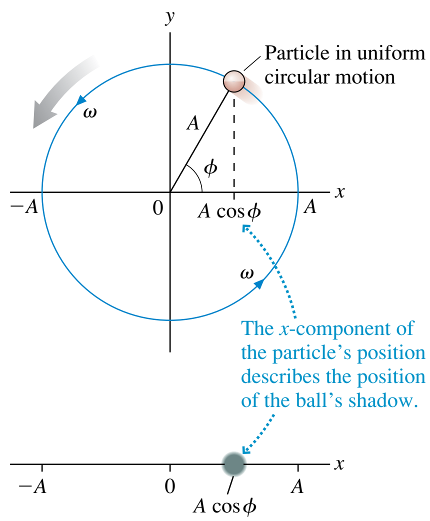
$$\phi = \omega t \tag{14.11}$$

As ϕ increases, the particle's x-component is

$$x(t) = A\cos\omega t \tag{14.12}$$

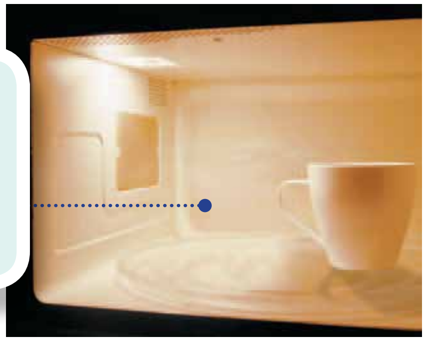
This is identical to Equation 14.5 for the position of a mass on a spring! Thus the x-component of a particle in uniform circular motion is simple harmonic motion.

FIGURE 14.7 A particle in uniform circular motion with radius A and angular velocity ω.



NOTE ▶ When used to describe oscillatory motion, ω is called the *angular frequency* rather than the angular velocity. The angular frequency of an oscillator has the same numerical value, in rad/s, as the angular velocity of the corresponding particle in circular motion. ◀

Analogy is used throughout the text and figures to consolidate student understanding by comparing with a more familiar concept or situation.



A cup on the turntable in a microwave oven moves in a circle. But from the outside, you see the cup sliding back and forth—in simple harmonic motion!

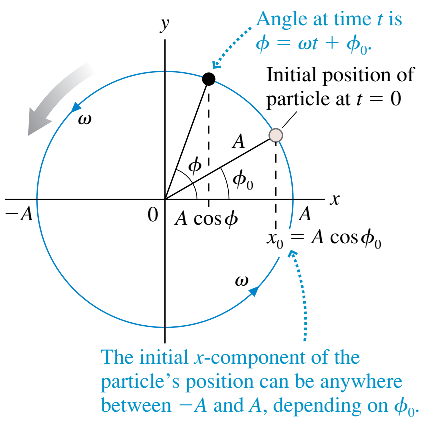
The names and units can be a bit confusing until you get used to them. It may help to notice that *cycle* and *oscillation* are not true units. Unlike the "standard meter" or the "standard kilogram," to which you could compare a length or a mass, there is no "standard cycle" to which you can compare an oscillation. Cycles and oscillations are simply counted events. Thus the frequency f has units of hertz, where $1\text{ Hz} = 1\text{ s}^{-1}$. We may *say* "cycles per second" just to be clear, but the actual units are only "per second."

The radian is the SI unit of angle. However, the radian is a *defined* unit. Further, its definition as a ratio of two lengths ($\theta = s/r$) makes it a pure number without dimensions. As we noted in Chapter 4, the unit of angle, be it radians or degrees, is really just a *name* to remind us that we're dealing with an angle. The 2π in the equation $\omega = 2\pi f$ (and in similar situations), which is stated without units, *means* 2π rad/cycle. When multiplied by the frequency f in cycles/s, it gives the frequency in rad/s. That is why, in this context, ω is called the angular *frequency*.

NOTE ▶ *Hertz* is specifically "cycles per second" or "oscillations per second." It is used for f but *not* for ω. We'll always be careful to use rad/s for ω, but you should be aware that many books give the units of ω as simply s^{-1}. ◀

The Phase Constant

FIGURE 14.8 A particle in uniform circular motion with initial angle ϕ_0.



Now we're ready to consider the issue of other initial conditions. The particle in Figure 14.7 started at $\phi_0 = 0$. This was equivalent to an oscillator starting at the far right edge, $x_0 = A$. **FIGURE 14.8** shows a more general situation in which the initial angle ϕ_0 can have any value. The angle at a later time t is then

$$\phi = \omega t + \phi_0 \tag{14.13}$$

In this case, the particle's projection onto the x-axis at time t is

$$x(t) = A\cos(\omega t + \phi_0) \tag{14.14}$$

If Equation 14.14 describes the particle's projection, then it must also be the position of an oscillator in simple harmonic motion. The oscillator's velocity v_x is found by taking the derivative dx/dt. The resulting equations,

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ v_x(t) &= -\omega A\sin(\omega t + \phi_0) = -v_{\max}\sin(\omega t + \phi_0) \end{aligned} \tag{14.15}$$

are the two primary kinematic equations of simple harmonic motion.

The quantity $\phi = \omega t + \phi_0$, which steadily increases with time, is called the **phase** of the oscillation. The phase is simply the *angle* of the circular-motion particle whose shadow matches the oscillator. The constant ϕ_0 is called the **phase constant.** It specifies the *initial conditions* of the oscillator.

To see what the phase constant means, set $t = 0$ in Equations 14.15:

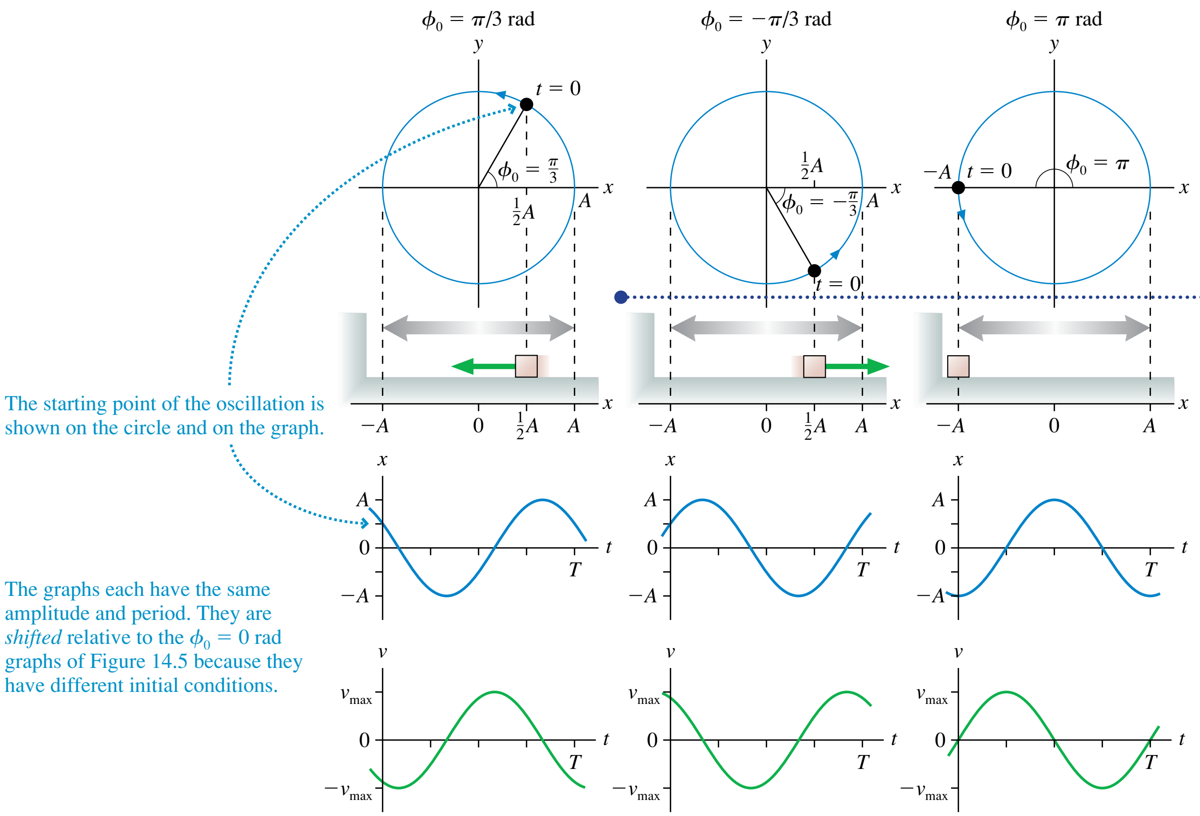
$$\begin{aligned} x_0 &= A\cos\phi_0 \\ v_{0x} &= -\omega A\sin\phi_0 \end{aligned} \tag{14.16}$$

The position x_0 and velocity v_{0x} at $t = 0$ are the initial conditions. **Different values of the phase constant correspond to different starting points on the circle and thus to different initial conditions.**

The perfect cosine function of Figure 14.5 and the equation $x(t) = A\cos\omega t$ are for an oscillation with $\phi_0 = 0$ rad. You can see from Equations 14.16 that $\phi_0 = 0$ rad implies $x_0 = A$ and $v_0 = 0$. That is, the particle starts from rest at the point of maximum displacement.

FIGURE 14.9 illustrates these ideas by looking at three values of the phase constant: $\phi_0 = \pi/3$ rad (60°), $-\pi/3$ rad (−60°), and π rad (180°). Notice that $\phi_0 = \pi/3$ rad and $\phi_0 = -\pi/3$ rad have the same starting position, $x_0 = \frac{1}{2}A$. This is a property of the cosine function in Equation 14.16. But these are *not* the same initial conditions. In one case the oscillator starts at $\frac{1}{2}A$ while moving to the right, in the other case it starts at $\frac{1}{2}A$ while moving to the left. You can distinguish between the two by visualizing the motion.

FIGURE 14.9 Oscillations described by the phase constants $\phi_0 = \pi/3$ rad, $-\pi/3$ rad, and π rad.



Extensive use is made of **multiple representations**—placing different representations side by side to help students develop the key skill of translating between words, math, and figures. Essential to good problem-solving, this skill is overlooked in most physics textbooks.

All values of the phase constant ϕ_0 between 0 and π rad correspond to a particle in the upper half of the circle and *moving to the left.* Thus v_{0x} is negative. All values of the phase constant ϕ_0 between π and 2π rad (or, as they are usually stated, between $-\pi$ and 0 rad) have the particle in the lower half of the circle and *moving to the right.* Thus v_{0x} is positive. If you're told that the oscillator is at $x = \frac{1}{2}A$ and moving to the right at $t = 0$, then the phase constant must be $\phi_0 = -\pi/3$ rad, not $+\pi/3$ rad.

EXAMPLE 14.4 Using the initial conditions

An object on a spring oscillates with a period of 0.80 s and an amplitude of 10 cm. At $t = 0$ s, it is 5.0 cm to the left of equilibrium and moving to the left. What are its position and direction of motion at $t = 2.0$ s?

MODEL An object oscillating on a spring is in simple harmonic motion.

SOLVE We can find the phase constant ϕ_0 from the initial condition $x_0 = -5.0\text{ cm} = A\cos\phi_0$. This condition gives

$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pm\frac{2}{3}\pi \text{ rad} = \pm 120°$$

Because the oscillator is moving to the *left* at $t = 0$, it is in the upper half of the circular-motion diagram and must have a phase constant between 0 and π rad. Thus ϕ_0 is $\frac{2}{3}\pi$ rad. The angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80\text{ s}} = 7.85\text{ rad/s}$$

Thus the object's position at time $t = 2.0$ s is

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ &= (10\text{ cm})\cos\left((7.85\text{ rad/s})(2.0\text{ s}) + \frac{2}{3}\pi\right) \\ &= (10\text{ cm})\cos(17.8\text{ rad}) = 5.0\text{ cm} \end{aligned}$$

The object is now 5.0 cm to the right of equilibrium. But which way is it moving? There are two ways to find out. The direct way is to calculate the velocity at $t = 2.0$ s:

$$v_x = -\omega A\sin(\omega t + \phi_0) = +68\text{ cm/s}$$

The velocity is positive, so the motion is to the right. Alternatively, we could note that the phase at $t = 2.0$ s is $\phi = 17.8$ rad. Dividing by π, you can see that

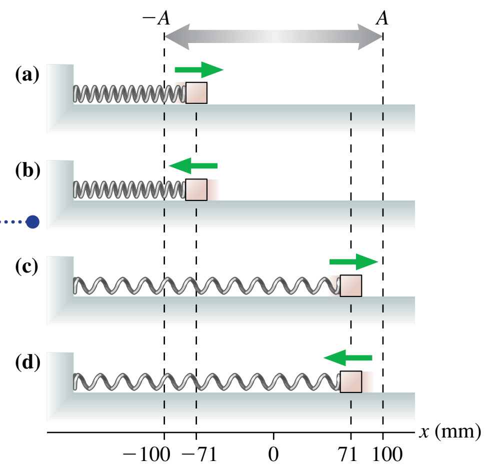
$$\phi = 17.8\text{ rad} = 5.67\pi\text{ rad} = (4\pi + 1.67\pi)\text{ rad}$$

The 4π rad represents two complete revolutions. The "extra" phase of 1.67π rad falls between π and 2π rad, so the particle in the circular-motion diagram is in the lower half of the circle and moving to the right.

Stop to Think questions at the end of a section allow students to quickly check their understanding. Using powerful ranking-task and graphical techniques, they are designed to efficiently probe key misconceptions and encourage active reading. (Answers are provided at the end of the chapter.)

NOTE ▶ The inverse-cosine function $\cos^{-1}$ is a *two-valued* function. Your calculator returns a single value, an angle between 0 rad and π rad. But the negative of this angle is also a solution. As Example 14.4 demonstrates, you must use additional information to choose between them. ◀

STOP TO THINK 14.2 The figure shows four oscillators at $t = 0$. Which one has the phase constant $\phi_0 = \pi/4$ rad?



14.3 Energy in Simple Harmonic Motion

We've begun to develop the mathematical language of simple harmonic motion, but thus far we haven't included any physics. We've made no mention of the mass of the object or the spring constant of the spring. An energy analysis, using the tools of Chapters 10 and 11, is a good starting place.

FIGURE 14.10a shows an object oscillating on a spring, our prototype of simple harmonic motion. Now we'll specify that the object has mass m, the spring has spring constant k, and the motion takes place on a frictionless surface. You learned in Chapter 10 that the elastic potential energy when the object is at position x is $U_s = \frac{1}{2}k(\Delta x)^2$, where $\Delta x = x - x_e$ is the displacement from the equilibrium position x_e. In this chapter we'll always use a coordinate system in which $x_e = 0$, making $\Delta x = x$. There's no chance for confusion with gravitational potential energy, so we can omit the subscript s and write the elastic potential energy as

$$U = \frac{1}{2}kx^2 \tag{14.17}$$

Thus the mechanical energy of an object oscillating on a spring is

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{14.18}$$

FIGURE 14.10b is an energy diagram, showing the potential-energy curve $U = \frac{1}{2}kx^2$ as a parabola. Recall that a particle oscillates between the *turning points* where the total energy line E crosses the potential-energy curve. The left turning point is at $x = -A$, and the right turning point is at $x = +A$. To go beyond these points would require a negative kinetic energy, which is physically impossible.

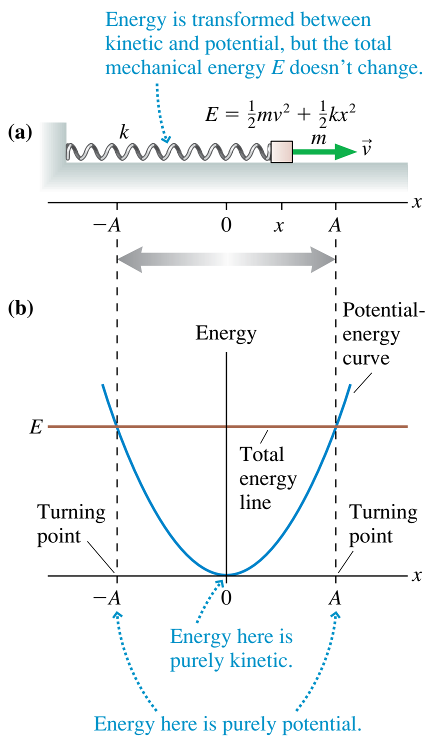
You can see that **the particle has purely potential energy at $x = \pm A$ and purely kinetic energy as it passes through the equilibrium point at $x = 0$.** At maximum displacement, with $x = \pm A$ and $v = 0$, the energy is

$$E(\text{at } x = \pm A) = U = \frac{1}{2}kA^2 \tag{14.19}$$

At $x = 0$, where $v = \pm v_{max}$, the energy is

$$E(\text{at } x = 0) = K = \frac{1}{2}m(v_{max})^2 \tag{14.20}$$

FIGURE 14.10 The energy is transformed between kinetic energy and potential energy as the object oscillates, but the mechanical energy $E = K + U$ doesn't change.



The system's mechanical energy is conserved because the surface is frictionless and there are no external forces, so the energy at maximum displacement and the energy at maximum speed, Equations 14.19 and 14.20, must be equal. That is

$$\frac{1}{2}m(v_{\max})^2 = \frac{1}{2}kA^2 \tag{14.21}$$

Thus the maximum speed is related to the amplitude by

$$v_{\max} = \sqrt{\frac{k}{m}}A \tag{14.22}$$

This is a relationship based on the physics of the situation.

Earlier, using kinematics, we found that

$$v_{\max} = \frac{2\pi A}{T} = 2\pi f A = \omega A \tag{14.23}$$

Comparing Equations 14.22 and 14.23, we see that frequency and period of an oscillating spring are determined by the spring constant k and the object's mass m:

$$\omega = \sqrt{\frac{k}{m}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}} \tag{14.24}$$

These three expressions are really only one equation. They say the same thing, but each expresses it in slightly different terms.

Equations 14.24 are the answer to the second question we posed at the beginning of the chapter, where we asked how the period and frequency are related to the object's mass m, the spring constant k, and the amplitude A. It is perhaps surprising, but **the period and frequency do not depend on the amplitude A.** A small oscillation and a large oscillation have the same period.

Because energy is conserved, we can combine Equations 14.18, 14.19, and 14.20 to write

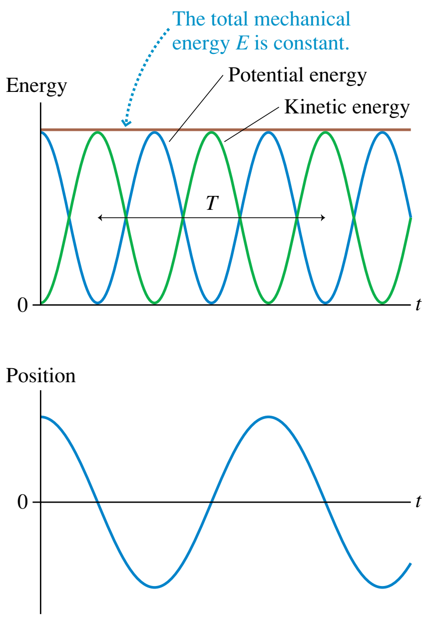
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}m(v_{\max})^2 \quad \text{(conservation of energy)} \tag{14.25}$$

Any pair of these expressions may be useful, depending on the known information. For example, you can use the amplitude A to find the speed at any point x by combining the first and second expressions for E. The speed v at position x is

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \omega\sqrt{A^2 - x^2} \tag{14.26}$$

FIGURE 14.11 shows graphically how the kinetic and potential energy change with time. They both oscillate but remain *positive* because x and v are squared. Energy is continuously being transformed back and forth between the kinetic energy of the moving block and the stored potential energy of the spring, but their sum remains constant. Notice that K and U both oscillate *twice* each period; make sure you understand why.

FIGURE 14.11 Kinetic energy, potential energy, and the total mechanical energy for simple harmonic motion.



EXAMPLE 14.5 Using conservation of energy

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s.

a. At what position or positions is the block's speed 1.0 m/s?
b. What is the spring constant?

MODEL The motion is SHM. Energy is conserved.

SOLVE a. The block starts from the point of maximum displacement, where $E = U = \frac{1}{2}kA^2$. At a later time, when the position is x and the speed is v, energy conservation requires

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Solving for x, we find

$$x = \sqrt{A^2 - \frac{mv^2}{k}} = \sqrt{A^2 - \left(\frac{v}{\omega}\right)^2}$$

where we used $k/m = \omega^2$ from Equation 14.24. The angular frequency is easily found from the period: $\omega = 2\pi/T = 7.85$ rad/s. Thus

$$x = \sqrt{(0.20\text{ m})^2 - \left(\frac{1.0\text{ m/s}}{7.85\text{ rad/s}}\right)^2} = \pm 0.15\text{ m} = \pm 15\text{ cm}$$

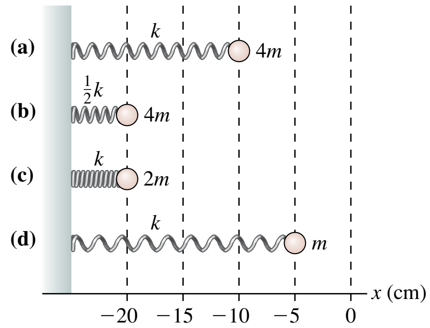
There are two positions because the block has this speed on either side of equilibrium.

b. Although part a did not require that we know the spring constant, it is straightforward to find from Equation 14.24:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2(0.50\text{ kg})}{(0.80\text{ s})^2} = 31\text{ N/m}$$

STOP TO THINK 14.3 The four springs shown here have been compressed from their equilibrium position at $x = 0$ cm. When released, the attached mass will start to oscillate. Rank in order, from highest to lowest, the maximum speeds of the masses.



14.4 The Dynamics of Simple Harmonic Motion

Our analysis thus far has been based on the experimental observation that the oscillation of a spring "looks" sinusoidal. It's time to show that Newton's second law *predicts* sinusoidal motion.

A motion diagram will help us visualize the object's acceleration. FIGURE 14.12 shows one cycle of the motion, separating motion to the left and motion to the right to make the diagram clear. As you can see, the object's velocity is large as it passes through the equilibrium point at $x = 0$, but $\vec{v}$ is *not changing* at that point. Acceleration measures the *change* of the velocity; hence $\vec{a} = \vec{0}$ at $x = 0$.

FIGURE 14.12 Motion diagram of simple harmonic motion. The left and right motions are separated vertically for clarity but really occur along the same line.

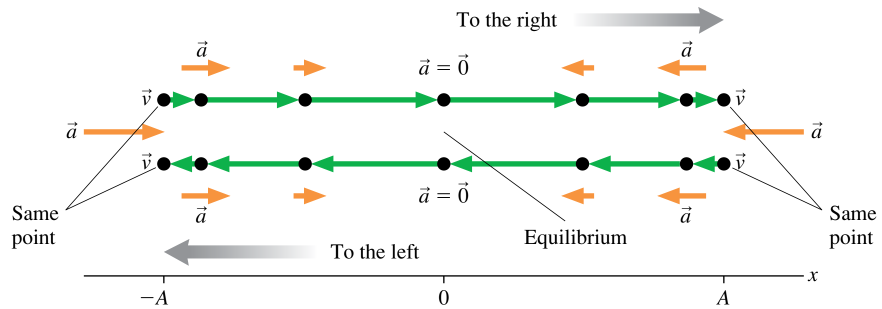
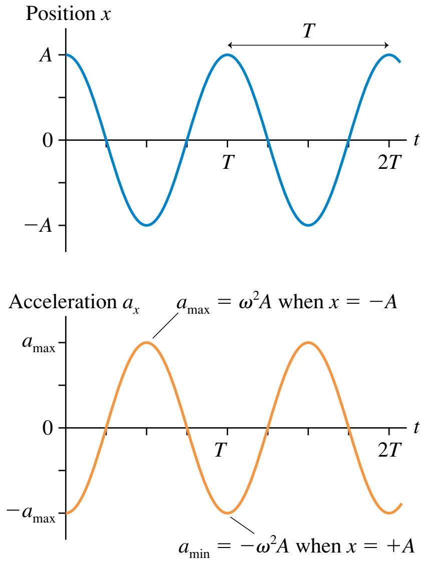


FIGURE 14.13 Position and acceleration graphs for an oscillating spring. We've chosen $\phi_0 = 0$.



In contrast, the velocity is changing rapidly at the turning points. At the right turning point, $\vec{v}$ changes from a right-pointing vector to a left-pointing vector. Thus the acceleration $\vec{a}$ at the right turning point is large and *to the left.* In one-dimensional motion, the acceleration component a_x has a large *negative* value at the right turning point. Similarly, the acceleration $\vec{a}$ at the left turning point is large and *to the right.* Consequently, a_x has a large positive value at the left turning point.

Our motion-diagram analysis suggests that the acceleration a_x is most positive when the displacement is most negative, most negative when the displacement is a maximum, and zero when $x = 0$. This is confirmed by taking the derivative of the velocity:

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-\omega A \sin \omega t) = -\omega^2 A \cos \omega t \tag{14.27}$$

then graphing it.

FIGURE 14.13 shows the position graph that we started with in Figure 14.4 and the corresponding acceleration graph. Comparing the two, you can see that the acceleration graph looks like an upside-down position graph. In fact, because $x = A\cos\omega t$, Equation 14.27 for the acceleration can be written

$$a_x = -\omega^2 x \tag{14.28}$$

That is, **the acceleration is proportional to the negative of the displacement.** The acceleration is, indeed, most positive when the displacement is most negative and is most negative when the displacement is most positive.

Recall that the acceleration is related to the net force by Newton's second law. Consider again our prototype mass on a spring, shown in **FIGURE 14.14**. This is the simplest possible oscillation, with no distractions due to friction or gravitational forces. We will assume the spring itself to be massless.

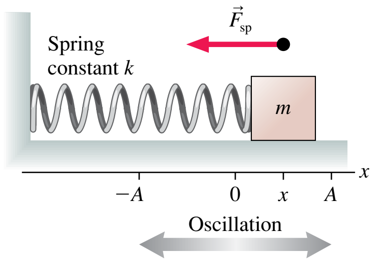


FIGURE 14.14 The prototype of simple harmonic motion: a mass oscillating on a horizontal spring without friction.

As you learned in Chapter 10, the spring force is given by Hooke's law:

$$(F_{sp})_x = -k\,\Delta x \tag{14.29}$$

The minus sign indicates that the spring force is a **restoring force,** a force that always points back toward the equilibrium position. If we place the origin of the coordinate system at the equilibrium position, as we've done throughout this chapter, then $\Delta x = x$ and Hooke's law is simply $(F_{sp})_x = -kx$.

The x-component of Newton's second law for the object attached to the spring is

$$(F_{net})_x = (F_{sp})_x = -kx = ma_x \tag{14.30}$$

Equation 14.30 is easily rearranged to read

$$a_x = -\frac{k}{m}x \tag{14.31}$$

You can see that Equation 14.31 is identical to Equation 14.28 if the system oscillates with angular frequency $\omega = \sqrt{k/m}$. We previously found this expression for ω from an energy analysis. Our experimental observation that the acceleration is proportional to the *negative* of the displacement is exactly what Hooke's law would lead us to expect. That's the good news.

The bad news is that a_x is not a constant. As the object's position changes, so does the acceleration. Nearly all of our kinematic tools have been based on constant acceleration. We can't use those tools to analyze oscillations, so we must go back to the very definition of acceleration:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

Acceleration is the second derivative of position with respect to time. If we use this definition in Equation 14.31, it becomes

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{(equation of motion for a mass on a spring)} \tag{14.32}$$

Equation 14.32, which is called the **equation of motion,** is a second-order differential equation. Unlike other equations we've dealt with, Equation 14.32 cannot be solved by direct integration. We'll need to take a different approach.

Solving the Equation of Motion

The solution to an algebraic equation such as $x^2 = 4$ is a number. The solution to a differential equation is a *function.* The x in Equation 14.32 is really $x(t)$, the position as a function of time. The solution to this equation is a function $x(t)$ whose second derivative is the function itself multiplied by $(-k/m)$.

One important property of differential equations that you will learn about in math is that the solutions are *unique.* That is, there is only *one* solution to Equation 14.32 that satisfies the initial conditions. If we were able to *guess* a solution, the uniqueness property would tell us that we had found the *only* solution. That might seem a rather

An optical technique called *interferometry* reveals the bell-like vibrations of a wine glass.

strange way to solve equations, but in fact differential equations are frequently solved by using your knowledge of what the solution needs to look like to guess an appropriate function. Let us give it a try!

We know from experimental evidence that the oscillatory motion of a spring appears to be sinusoidal. Let us *guess* that the solution to Equation 14.32 should have the functional form

$$x(t) = A\cos(\omega t + \phi_0) \tag{14.33}$$

where A, ω, and ϕ_0 are unspecified constants that we can adjust to any values that might be necessary to satisfy the differential equation.

If you were to guess that a solution to the algebraic equation $x^2 = 4$ is $x = 2$, you would verify your guess by substituting it into the original equation to see if it works. We need to do the same thing here: Substitute our guess for $x(t)$ into Equation 14.32 to see if, for an appropriate choice of the three constants, it works. To do so, we need the second derivative of $x(t)$. That is straightforward:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ \frac{dx}{dt} &= -\omega A\sin(\omega t + \phi_0) \\ \frac{d^2x}{dt^2} &= -\omega^2 A\cos(\omega t + \phi_0) \end{aligned} \tag{14.34}$$

If we now substitute the first and third of Equations 14.34 into Equation 14.32, we find

$$-\omega^2 A\cos(\omega t + \phi_0) = -\frac{k}{m}A\cos(\omega t + \phi_0) \tag{14.35}$$

Equation 14.35 will be true at all instants of time if and only if $\omega^2 = k/m$. There do not seem to be any restrictions on the two constants A and ϕ_0—they are determined by the initial conditions.

So we have found—by guessing!—that *the* solution to the equation of motion for a mass oscillating on a spring is

$$x(t) = A\cos(\omega t + \phi_0) \tag{14.36}$$

where the angular frequency

$$\omega = 2\pi f = \sqrt{\frac{k}{m}} \tag{14.37}$$

is determined by the mass and the spring constant.

NOTE ▶ Once again we see that the oscillation frequency is independent of the amplitude A. ◀

Equations 14.36 and 14.37 seem somewhat anticlimactic because we've been using these results for the last several pages. But keep in mind that we had been *assuming* $x = A\cos\omega t$ simply because the experimental observations "looked" like a cosine function. We've now justified that assumption by showing that Equation 14.36 really is the solution to Newton's second law for a mass on a spring. **The *theory* of oscillation, based on Hooke's law for a spring and Newton's second law, is in good agreement with the experimental observations.** This conclusion gives an affirmative answer to the last of the three questions that we asked early in the chapter, which was whether the sinusoidal oscillation of SHM is a consequence of Newton's laws.

EXAMPLE 14.6 Analyzing an oscillator

At $t = 0$ s, a 500 g block oscillating on a spring is observed moving to the right at $x = 15$ cm. It reaches a maximum displacement of 25 cm at $t = 0.30$ s.

a. Draw a position-versus-time graph for one cycle of the motion.
b. At what times during the first cycle does the mass pass through $x = 20$ cm?

MODEL The motion is simple harmonic motion.

SOLVE a. The position equation of the block is $x(t) = A\cos(\omega t + \phi_0)$. We know that the amplitude is $A = 0.25$ m and that $x_0 = 0.15$ m. From these two pieces of information we obtain the phase constant:

$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}(0.60) = \pm 0.927 \text{ rad}$$

The object is initially moving to the right, which tells us that the phase constant must be between $-\pi$ and 0 rad. Thus $\phi_0 = -0.927$ rad. The block reaches its maximum displacement $x_{max} = A$ at time $t = 0.30$ s. At that instant of time

$$x_{max} = A = A\cos(\omega t + \phi_0)$$

This can be true only if $\cos(\omega t + \phi_0) = 1$, which requires $\omega t + \phi_0 = 0$. Thus

$$\omega = \frac{-\phi_0}{t} = \frac{-(-0.927 \text{ rad})}{0.30 \text{ s}} = 3.09 \text{ rad/s}$$

Now that we know ω, it is straightforward to compute the period:

$$T = \frac{2\pi}{\omega} = 2.0 \text{ s}$$

FIGURE 14.15 graphs $x(t) = (25 \text{ cm})\cos(3.09t - 0.927)$, where t is in s, from $t = 0$ s to $t = 2.0$ s.

b. From $x = A\cos(\omega t + \phi_0)$, the time at which the mass reaches position $x = 20$ cm is

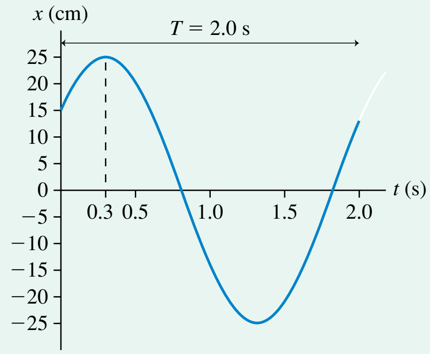


FIGURE 14.15 Position-versus-time graph for the oscillator of Example 14.6.

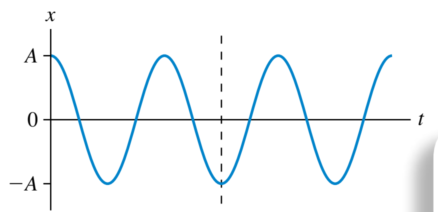
$$t = \frac{1}{\omega}\left(\cos^{-1}\left(\frac{x}{A}\right) - \phi_0\right)$$

$$= \frac{1}{3.09 \text{ rad/s}}\left(\cos^{-1}\left(\frac{20 \text{ cm}}{25 \text{ cm}}\right) + 0.927 \text{ rad}\right) = 0.51 \text{ s}$$

A calculator returns only one value of $\cos^{-1}$, in the range 0 to π rad, but we noted earlier that $\cos^{-1}$ actually has two values. Indeed, you can see in Figure 14.15 that there are two times at which the mass passes $x = 20$ cm. Because they are symmetrical on either side of $t = 0.30$ s, when $x = A$, the first point is $(0.51 \text{ s} - 0.30 \text{ s}) = 0.21$ s *before* the maximum. Thus the mass passes through $x = 20$ cm at $t = 0.09$ s and again at $t = 0.51$ s.

STOP TO THINK 14.4 This is the position graph of a mass on a spring. What can you say about the velocity and the force at the instant indicated by the dashed line?

a. Velocity is positive; force is to the right.
b. Velocity is negative; force is to the right.
c. Velocity is zero; force is to the right.
d. Velocity is positive; force is to the left.
e. Velocity is negative; force is to the left.
f. Velocity is zero; force is to the left.
g. Velocity and force are both zero.



14.5 Vertical Oscillations

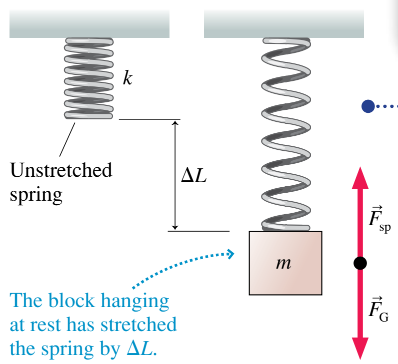
We have focused our analysis on a horizontally oscillating spring. But the typical demonstration you'll see in class is a mass bobbing up and down on a spring hung vertically from a support. Is it safe to assume that a vertical oscillation has the same mathematical description as a horizontal oscillation? Or does the additional force of gravity change the motion? Let us look at this more carefully.

FIGURE 14.16 shows a block of mass m hanging from a spring of spring constant k. An important fact to notice is that the equilibrium position of the block is *not* where the spring is at its unstretched length. At the equilibrium position of the block, where it hangs motionless, the spring has stretched by ΔL.

Finding ΔL is a static-equilibrium problem in which the upward spring force balances the downward gravitational force on the block. The y-component of the spring force is given by Hooke's law:

$$(F_{sp})_y = -k\,\Delta y = +k\,\Delta L \tag{14.38}$$

FIGURE 14.16 Gravity stretches the spring.



Extensive use is made of **multiple representations**—placing different representations side by side to help students develop the key skill of translating between words, math, and figures. Essential to good problem-solving, this skill is overlooked in most physics textbooks.

Equation 14.38 makes a distinction between ΔL, which is simply a *distance* and is a positive number, and the displacement Δy. The block is displaced downward, so $\Delta y = -\Delta L$. Newton's first law for the block in equilibrium is

$$(F_{\text{net}})_y = (F_{\text{sp}})_y + (F_G)_y = k\,\Delta L - mg = 0 \tag{14.39}$$

from which we can find

$$\Delta L = \frac{mg}{k} \tag{14.40}$$

This is the distance the spring stretches when the block is attached to it.

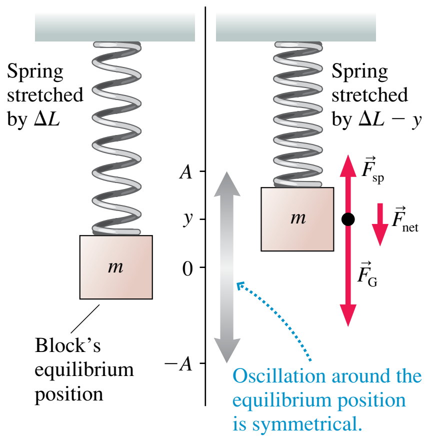


FIGURE 14.17 The block oscillates around the equilibrium position.

Let the block oscillate around this equilibrium position, as shown in **FIGURE 14.17**. We've now placed the origin of the y-axis at the block's equilibrium position in order to be consistent with our analyses of oscillations throughout this chapter. If the block moves upward, as the figure shows, the spring gets shorter compared to its equilibrium length, but the spring is still *stretched* compared to its unstretched length in Figure 14.16. When the block is at position y, the spring is stretched by an amount $\Delta L - y$ and hence exerts an *upward* spring force $F_{\text{sp}} = k(\Delta L - y)$. The net force on the block at this point is

$$(F_{\text{net}})_y = (F_{\text{sp}})_y + (F_G)_y = k(\Delta L - y) - mg = (k\,\Delta L - mg) - ky \tag{14.41}$$

But $k\,\Delta L - mg$ is zero, from Equation 14.40, so the net force on the block is simply

$$(F_{\text{net}})_y = -ky \tag{14.42}$$

Equation 14.42 for vertical oscillations is *exactly* the same as Equation 14.30 for horizontal oscillations, where we found $(F_{\text{net}})_x = -kx$. That is, the restoring force for vertical oscillations is identical to the restoring force for horizontal oscillations. The role of gravity is to determine where the equilibrium position is, but it doesn't affect the oscillatory motion around the equilibrium position.

Because the net force is the same, Newton's second law has exactly the same oscillatory solution:

$$y(t) = A\cos(\omega t + \phi_0) \tag{14.43}$$

with, again, $\omega = \sqrt{k/m}$. **The vertical oscillations of a mass on a spring are the same simple harmonic motion as those of a block on a horizontal spring.** This is an important finding because it was not obvious that the motion would still be simple harmonic motion when gravity was included.

Hand-drawn sketches are incorporated into select worked examples to provide a clear model of what students should draw during their own problem solving.

EXAMPLE 14.7 Bungee oscillations

An 83 kg student hangs from a bungee cord with spring constant 270 N/m. The student is pulled down to a point where the cord is 5.0 m longer than its unstretched length, then released. Where is the student, and what is his velocity 2.0 s later?

MODEL A bungee cord can be modeled as a spring. Vertical oscillations on the bungee cord are SHM.

VISUALIZE **FIGURE 14.18** shows the situation.

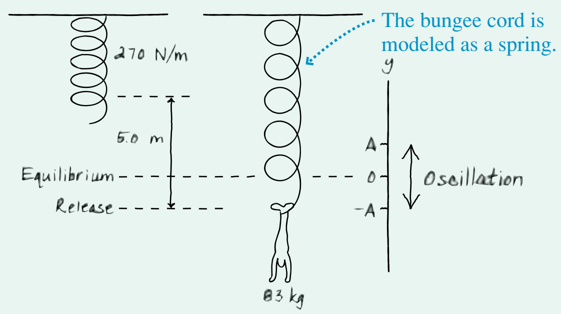


FIGURE 14.18 A student on a bungee cord oscillates about the equilibrium position.

SOLVE Although the cord is stretched by 5.0 m when the student is released, this is *not* the amplitude of the oscillation. Oscillations occur around the equilibrium position, so we have to begin by finding the equilibrium point where the student hangs motionless. The cord stretch at equilibrium is given by Equation 14.40:

$$\Delta L = \frac{mg}{k} = 3.0\text{ m}$$

Stretching the cord 5.0 m pulls the student 2.0 m below the equilibrium point, so $A = 2.0$ m. That is, the student oscillates with amplitude $A = 2.0$ m about a point 3.0 m beneath the bungee cord's original end point. The student's position as a function of time, as measured from the equilibrium position, is

$$y(t) = (2.0\text{ m})\cos(\omega t + \phi_0)$$

where $\omega = \sqrt{k/m} = 1.80$ rad/s. The initial condition

$$y_0 = A\cos\phi_0 = -A$$

requires the phase constant to be $\phi_0 = \pi$ rad. At $t = 2.0$ s the student's position and velocity are

$$y = (2.0\text{ m})\cos\big((1.80\text{ rad/s})(2.0\text{ s}) + \pi\text{ rad}\big) = 1.8\text{ m}$$
$$v_y = -\omega A\sin(\omega t + \phi_0) = -1.6\text{ m/s}$$

The student is 1.8 m *above* the equilibrium position, or 1.2m *below* the original end of the cord. Because his velocity is negative, he's passed through the highest point and is heading down.

14.6 The Pendulum

Now let's look at another very common oscillator: a pendulum. **FIGURE 14.19a** shows a mass m attached to a string of length L and free to swing back and forth. The pendulum's position can be described by the arc of length s, which is zero when the pendulum hangs straight down. Because angles are measured ccw, s and θ are positive when the pendulum is to the right of center, negative when it is to the left.

Two forces are acting on the mass: the string tension $\vec{T}$ and gravity $\vec{F}_G$. It will be convenient to repeat what we did in our study of circular motion: Divide the forces into tangential components, parallel to the motion, and radial components parallel to the string. These are shown on the free-body diagram of **FIGURE 14.19b**.

Newton's second law for the tangential component, parallel to the motion, is

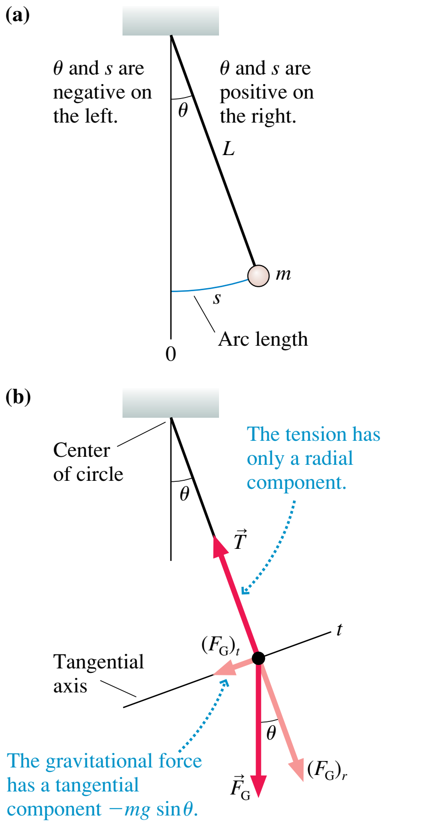
$$(F_{\text{net}})_t = \sum F_t = (F_G)_t = -mg\sin\theta = ma_t \qquad (14.44)$$

Using $a_t = d^2s/dt^2$ for acceleration "around" the circle, and noting that the mass cancels, we can write Equation 14.44 as

$$\frac{d^2s}{dt^2} = -g\sin\theta \qquad (14.45)$$

This is the equation of motion for an oscillating pendulum. The sine function makes this equation more complicated than the equation of motion for an oscillating spring.

FIGURE 14.19 The motion of a pendulum.



The Small-Angle Approximation

Suppose we restrict the pendulum's oscillations to *small angles* of less than about 10°. This restriction allows us to make use of an interesting and important piece of geometry.

FIGURE 14.20 shows an angle θ and a circular arc of length $s = r\theta$. A right triangle has been constructed by dropping a perpendicular from the top of the arc to the axis. The height of the triangle is $h = r\sin\theta$. Suppose that the angle θ is "small." In that case there is very little difference between h and s. If $h \approx s$, then $r\sin\theta \approx r\theta$. It follows that

$$\sin\theta \approx \theta \quad (\theta \text{ in radians})$$

The result that $\sin\theta \approx \theta$ for small angles is called the **small-angle approximation.** We can similarly note that $l \approx r$ for small angles. Because $l = r\cos\theta$, it follows that $\cos\theta \approx 1$. Finally, we can take the ratio of sine and cosine to find $\tan\theta \approx \sin\theta \approx \theta$. Table 14.3 summarizes the small-angle approximation. We will have other occasions to use the small-angle approximation throughout the remainder of this text.

NOTE ▶ The small-angle approximation is valid *only* if angle θ is in radians! ◀

How small does θ have to be to justify using the small-angle approximation? It's easy to use your calculator to find that the small-angle approximation is good to three

FIGURE 14.20 The geometrical basis of the small-angle approximation.

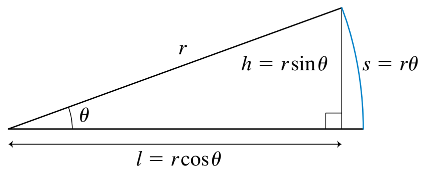


TABLE 14.3 Small-angle approximations. θ must be in radians.

$\sin\theta \approx \theta$	$\tan\theta \approx \sin\theta \approx \theta$
$\cos\theta \approx 1$	

The pendulum clock has been used for hundreds of years.

significant figures, an error of $\leq 0.1\%$, up to angles of ≈ 0.10 rad ($\approx 5°$). In practice, we will use the approximation up to about 10°, but for angles any larger it rapidly loses validity and produces unacceptable results.

If we restrict the pendulum to $\theta < 10°$, we can use $\sin\theta \approx \theta$. In that case, Equation 14.44 for the net force on the mass is

$$(F_{\text{net}})_t = -mg \sin\theta \approx -mg\theta = -\frac{mg}{L}s$$

where, in the last step, we used the fact that angle θ is related to the arc length by $\theta = s/L$. Then the equation of motion becomes

$$\frac{d^2 s}{dt^2} = -\frac{g}{L}s \tag{14.46}$$

This is *exactly* the same as Equation 14.32 for a mass oscillating on a spring. The names are different, with x replaced by s and k/m by g/L, but that does not make it a different equation.

Because we know the solution to the spring problem, we can immediately write the solution to the pendulum problem just by changing variables and constants:

$$s(t) = A\cos(\omega t + \phi_0) \qquad \text{or} \qquad \theta(t) = \theta_{\max}\cos(\omega t + \phi_0) \tag{14.47}$$

The angular frequency

$$\omega = 2\pi f = \sqrt{\frac{g}{L}} \tag{14.48}$$

is determined by the length of the string. The pendulum is interesting in that **the frequency, and hence the period, is independent of the mass.** It depends only on the length of the pendulum. The amplitude A and the phase constant ϕ_0 are determined by the initial conditions, just as they were for an oscillating spring.

EXAMPLE 14.8 The maximum angle of a pendulum

A 300 g mass on a 30-cm-long string oscillates as a pendulum. It has a speed of 0.25 m/s as it passes through the lowest point. What maximum angle does the pendulum reach?

MODEL Assume that the angle remains small, in which case the motion is simple harmonic motion.

SOLVE The angular frequency of the pendulum is

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\text{ m/s}^2}{0.30\text{ m}}} = 5.72\text{ rad/s}$$

The speed at the lowest point is $v_{\max} = \omega A$, so the amplitude is

$$A = s_{\max} = \frac{v_{\max}}{\omega} = \frac{0.25\text{ m/s}}{5.72\text{ rad/s}} = 0.0437\text{ m}$$

The maximum angle, at the maximum arc length $s_{\max}$, is

$$\theta_{\max} = \frac{s_{\max}}{L} = \frac{0.0437\text{ m}}{0.30\text{ m}} = 0.146\text{ rad} = 8.3°$$

ASSESS Because the maximum angle is less than 10°, our analysis based on the small-angle approximation is reasonable.

Data-based Examples help students with the skill of drawing conclusions from laboratory data. Designed to supplement lab-based instruction, these examples also help students in general with mathematical reasoning, graphical interpretation, and assessment of results.

EXAMPLE 14.9 The gravimeter

Deposits of minerals and ore can alter the local value of the free-fall acceleration because they tend to be denser than surrounding rocks. Geologists use a *gravimeter*—an instrument that accurately measures the local free-fall acceleration—to search for ore deposits. One of the simplest gravimeters is a pendulum. To achieve the highest accuracy, a stopwatch is used to time 100 oscillations of a pendulum of different lengths. At one location in the field, a geologist makes the following measurements:

Length (m)	Time (s)
0.500	141.7
1.000	200.6
1.500	245.8
2.000	283.5

What is the local value of g?

MODEL Assume the oscillation angle is small, in which case the motion is simple harmonic motion with a period independent of the mass of the pendulum. Because the data are known to four significant figures (± 1 mm on the length and ± 0.1 s on the timing, both of which are easily achievable), we expect to determine g to four significant figures.

SOLVE From Equation 14.48, using $f = 1/T$, we find

$$T^2 = \left(2\pi\sqrt{\frac{L}{g}}\right)^2 = \frac{4\pi^2}{g}L$$

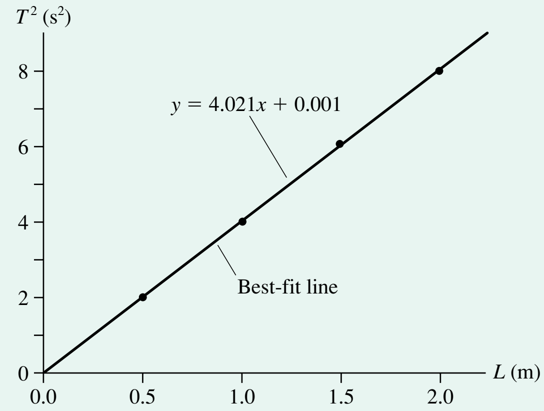
That is, the square of a pendulum's period is proportional to its length. Consequently, a graph of T^2 versus L should be a straight line passing through the origin with slope $4\pi^2/g$. We can use the experimentally measured slope to determine g. **FIGURE 14.21** is a graph of the data, with the period found by dividing the measured time by 100.

As expected, the graph is a straight line passing through the origin. The slope of the best-fit line is 4.021 s^2/m. Consequently,

$$g = \frac{4\pi^2}{\text{slope}} = \frac{4\pi^2}{4.021\ \text{s}^2/\text{m}} = 9.818\ \text{m/s}^2$$

ASSESS The fact that the graph is linear and passes through the origin confirms our model of the situation. Had this *not* been the case, we would have had to conclude either that our model of the pendulum as a simple, small-angle pendulum was not valid or that our measurements were bad. This is an important reason for having multiple data points rather than using only one length.

FIGURE 14.21 Graph of the square of the pendulum's period versus its length.



The Conditions for Simple Harmonic Motion

You can begin to see how, in a sense, we have solved *all* simple-harmonic-motion problems once we have solved the problem of the horizontal spring. The restoring force of a spring, $F_{sp} = -kx$, is directly proportional to the displacement x from equilibrium. The pendulum's restoring force, in the small-angle approximation, is directly proportional to the displacement s. A restoring force that is directly proportional to the displacement from equilibrium is called a **linear restoring force.** For *any* linear restoring force, the equation of motion is identical to the spring equation (other than perhaps using different symbols). Consequently, **any system with a linear restoring force will undergo simple harmonic motion around the equilibrium position.**

This is why an oscillating spring is the prototype of SHM. Everything that we learn about an oscillating spring can be applied to the oscillations of any other linear restoring force, ranging from the vibration of airplane wings to the motion of electrons in electric circuits. Let's summarize this information with a Tactics Box.

TACTICS BOX 14.1 **Identifying and analyzing simple harmonic motion**

1. If the net force acting on a particle is a linear restoring force, the motion will be simple harmonic motion around the equilibrium position.
2. The position as a function of time is $x(t) = A\cos(\omega t + \phi_0)$. The velocity as a function of time is $v_x(t) = -\omega A\sin(\omega t + \phi_0)$. The maximum speed is $v_{max} = \omega A$. The equations are given here in terms of x, but they can be written in terms of y, θ, or some other parameter if the situation calls for it.
3. The amplitude A and the phase constant ϕ_0 are determined by the initial conditions through $x_0 = A\cos\phi_0$ and $v_{0x} = -\omega A\sin\phi_0$.
4. The angular frequency ω (and hence the period $T = 2\pi/\omega$) depends on the physics of the particular situation. But ω does *not* depend on A or ϕ_0.
5. Mechanical energy is conserved. Thus $\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}m(v_{max})^2$. Energy conservation provides a relationship between position and velocity that is independent of time.

Exercises 7–12, 15–19

Tactics Boxes give step-by-step procedures for developing specific skills (drawing free-body diagrams, using ray tracing, etc.).

The Physical Pendulum

A mass on a string is often called a *simple pendulum.* But you can also make a pendulum from any solid object that swings back and forth on a pivot under the influence of gravity. This is called a *physical pendulum.*

FIGURE 14.22 shows a physical pendulum of mass M for which the distance between the pivot and the center of mass is l. The moment arm of the gravitational force acting at the center of mass is $d = l \sin\theta$, so the gravitational torque is

$$\tau = -Mgd = -Mgl \sin\theta$$

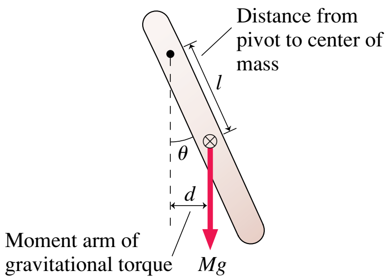


FIGURE 14.22 A physical pendulum.

The torque is negative because, for positive θ, it's causing a clockwise rotation. If we restrict the angle to being small ($\theta < 10°$), as we did for the simple pendulum, we can use the small-angle approximation to write

$$\tau = -Mgl\theta \tag{14.49}$$

Gravity causes a linear restoring torque on the pendulum—that is, the torque is directly proportional to the angular displacement θ—so we expect the physical pendulum to undergo SHM.

From Chapter 12, Newton's second law for rotational motion is

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{\tau}{I}$$

where I is the object's moment of inertia about the pivot point. Using Equation 14.49 for the torque, we find

$$\frac{d^2\theta}{dt^2} = \frac{-Mgl}{I}\theta \tag{14.50}$$

Comparison with Equation 14.32 shows that this is again the SHM equation of motion, this time with angular frequency

$$\omega = 2\pi f = \sqrt{\frac{Mgl}{I}} \tag{14.51}$$

It appears that the frequency depends on the mass of the pendulum, but recall that the moment of inertia is directly proportional to M. Thus M cancels and the frequency of a physical pendulum, like that of a simple pendulum, is independent of mass.

EXAMPLE 14.10 A swinging leg as a pendulum

A student in a biomechanics lab measures the length of his leg, from hip to heel, to be 0.90 m. What is the frequency of the pendulum motion of the student's leg? What is the period?

MODEL We can model a human leg reasonably well as a rod of uniform cross section, pivoted at one end (the hip) to form a physical pendulum. The center of mass of a uniform leg is at the midpoint, so $l = L/2$.

SOLVE The moment of inertia of a rod pivoted about one end is $I = \frac{1}{3}ML^2$, so the pendulum frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{Mgl}{I}} = \frac{1}{2\pi}\sqrt{\frac{Mg(L/2)}{ML^2/3}} = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}} = 0.64 \text{ Hz}$$

The corresponding period is $T = 1/f = 1.6$ s. Notice that we didn't need to know the mass.

ASSESS As you walk, your legs do swing as physical pendulums as you bring them forward. The frequency is fixed by the length of your legs and their distribution of mass; it doesn't depend on amplitude. Consequently, you don't increase your walking speed by taking more rapid steps—changing the frequency is difficult. You simply take longer strides, changing the amplitude but not the frequency.

Life-science and bioengineering worked examples and applications focus on the physics of life-science situations in order to serve the needs of life-science students taking a calculus-based physics class.

STOP TO THINK 14.5 One person swings on a swing and finds that the period is 3.0 s. A second person of equal mass joins him. With two people swinging, the period is

a. 6.0 s
b. >3.0 s but not necessarily 6.0 s
c. 3.0 s
d. <3.0 s but not necessarily 1.5 s
e. 1.5 s
f. Can't tell without knowing the length

14.7 Damped Oscillations

A pendulum left to itself gradually slows down and stops. The sound of a ringing bell gradually dies away. All real oscillators do run down—some very slowly but others quite quickly—as friction or other dissipative forces transform their mechanical energy into the thermal energy of the oscillator and its environment. An oscillation that runs down and stops is called a **damped oscillation.**

The shock absorbers in cars and trucks are heavily damped springs. The vehicle's vertical motion, after hitting a rock or a pothole, is a damped oscillation.

There are many possible reasons for the dissipation of energy, such as air resistance, friction, and internal forces within a metal spring as it flexes. The forces involved in dissipation are complex, but a simple *linear drag* model gives a quite accurate description of most damped oscillations. That is, we'll assume a drag force that depends linearly on the velocity as

$$\vec{D} = -b\vec{v} \quad \text{(model of the drag force)} \qquad (14.52)$$

where the minus sign is the mathematical statement that the force is always opposite in direction to the velocity in order to slow the object.

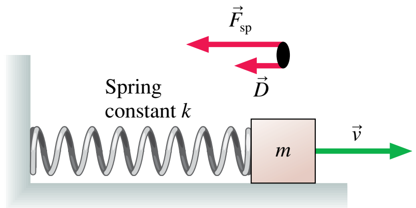
The **damping constant** b depends in a complicated way on the shape of the object *and* on the viscosity of the air or other medium in which the particle moves. The damping constant plays the same role in our model of air resistance that the coefficient of friction does in our model of friction.

The units of b need to be such that they will give units of force when multiplied by units of velocity. As you can confirm, these units are kg/s. A value $b = 0$ kg/s corresponds to the limiting case of no resistance, in which case the mechanical energy is conserved. A typical value of b for a spring or a pendulum in air is ≤ 0.10 kg/s. Objects moving in a liquid can have significantly larger values of b.

FIGURE 14.23 shows a mass oscillating on a spring in the presence of a drag force. With the drag included, Newton's second law is

$$(F_{net})_x = (F_{sp})_x + D_x = -kx - bv_x = ma_x \qquad (14.53)$$

FIGURE 14.23 An oscillating mass in the presence of a drag force.



Using $v_x = dx/dt$ and $a_x = d^2x/dt^2$, we can write Equation 14.53 as

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad (14.54)$$

Equation 14.54 is the equation of motion of a damped oscillator. If you compare it to Equation 14.32, the equation of motion for a block on a frictionless surface, you'll see that it differs by the inclusion of the term involving dx/dt.

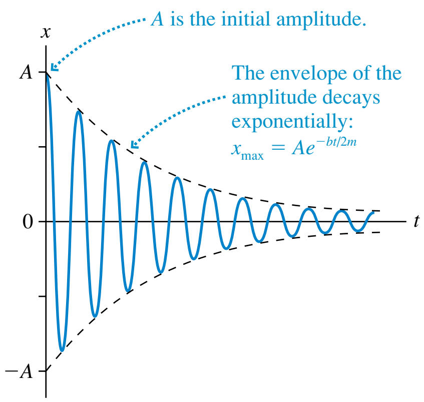
Equation 14.54 is another second-order differential equation. We will simply assert (and, as a homework problem, you can confirm) that the solution is

$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0) \quad \text{(damped oscillator)} \qquad (14.55)$$

where the angular frequency is given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0{}^2 - \frac{b^2}{4m^2}} \qquad (14.56)$$

FIGURE 14.24 Position-versus-time graph for a damped oscillator.

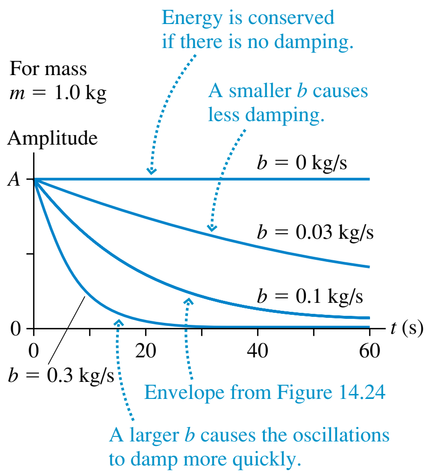


Here $\omega_0 = \sqrt{k/m}$ is the angular frequency of an undamped oscillator ($b = 0$). The constant e is the base of natural logarithms, so $e^{-bt/2m}$ is an *exponential function.*

Because $e^0 = 1$, Equation 14.55 reduces to our previous solution, $x(t) = A\cos(\omega t + \phi_0)$, when $b = 0$. This makes sense and gives us confidence in Equation 14.55. A *lightly damped* system, which oscillates many times before stopping, is one for which $b/2m \ll \omega_0$. In that case, $\omega \approx \omega_0$ is a good approximation. That is, light damping does not affect the oscillation frequency.

FIGURE 14.24 is a graph of the position $x(t)$ for a lightly damped oscillator, as given by Equation 14.55. Notice that the term $Ae^{-bt/2m}$, which is shown by the dashed line,

FIGURE 14.25 Several oscillation envelopes, corresponding to different values of the damping constant b.



acts as a slowly varying amplitude:

$$x_{\max}(t) = Ae^{-bt/2m} \tag{14.57}$$

where A is the *initial* amplitude, at $t = 0$. The oscillation keeps bumping up against this line, slowly dying out with time.

A slowly changing line that provides a border to a rapid oscillation is called the **envelope** of the oscillations. In this case, the oscillations have an *exponentially decaying envelope.* Make sure you study Figure 14.24 long enough to see how both the oscillations and the decaying amplitude are related to Equation 14.55.

Changing the amount of damping, by changing the value of b, affects how quickly the oscillations decay. **FIGURE 14.25** shows just the envelope $x_{\max}(t)$ for several oscillators that are identical except for the value of the damping constant b. (You need to imagine a rapid oscillation within each envelope, as in Figure 14.24.) Increasing b causes the oscillations to damp more quickly, while decreasing b makes them last longer.

MATHEMATICAL ASIDE **Exponential decay**

Exponential decay occurs in a vast number of physical systems of importance in science and engineering. Mechanical vibrations, electric circuits, and nuclear radioactivity all exhibit exponential decay.

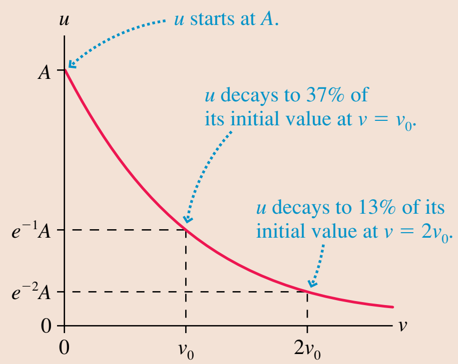
The number $e = 2.71828\ldots$ is the base of natural logarithms in the same way that 10 is the base of ordinary logarithms. It arises naturally in calculus from the integral

$$\int \frac{du}{u} = \ln u$$

This integral—which shows up in the analysis of many physical systems—frequently leads to solutions of the form

$$u = Ae^{-v/v_0} = A\exp(-v/v_0)$$

where exp is the *exponential function.*



A graph of u illustrates what we mean by *exponential decay.* It starts with $u = A$ at $v = 0$ (because $e^0 = 1$) and then steadily decays, asymptotically approaching zero. The quantity v_0 is called the *decay constant.* When $v = v_0$, $u = e^{-1}A = 0.37A$. When $v = 2v_0$, $u = e^{-2}A = 0.13A$.

Arguments of functions must be pure numbers, without units. That is, we can evaluate e^{-2}, but $e^{-2\text{ kg}}$ makes no sense. If v/v_0 is a pure number, which it must be, then the decay constant v_0 must have the same units as v. If v represents position, then v_0 is a length; if v represents time, then v_0 is a time interval. In a specific situation, v_0 is often called the *decay length* or the *decay time.* It is the length or time in which the quantity decays to 37% of its initial value.

No matter what the process is or what u represents, **a quantity that decays exponentially decays to 37% of its initial value when one decay constant has passed.** Thus exponential decay is a universal behavior. Every time you meet a new system that exhibits exponential decay, its behavior will be exactly the same as every other exponential decay. The decay curve always looks exactly like the figure shown here. Once you've learned the properties of exponential decay, you'll immediately know how to apply this knowledge to a new situation.

Energy in Damped Systems

When considering the oscillator's mechanical energy, it is useful to define the **time constant** τ (also called the *decay time*) to be

$$\tau = \frac{m}{b} \tag{14.58}$$

Because b has units of kg/s, τ has units of seconds. With this definition, we can write the oscillation amplitude as $x_{\max}(t) = Ae^{-t/2\tau}$.

Because of the drag force, the mechanical energy is no longer conserved. At any particular time we can compute the mechanical energy from

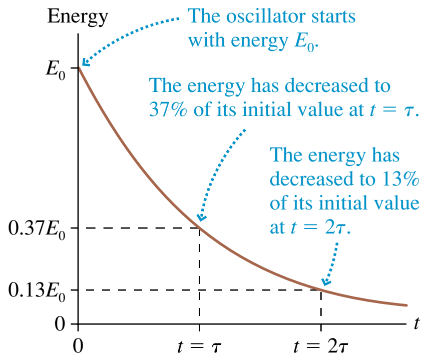
$$E(t) = \frac{1}{2}k(x_{\text{max}})^2 = \frac{1}{2}k(Ae^{-t/2\tau})^2 = \left(\frac{1}{2}kA^2\right)e^{-t/\tau} = E_0 e^{-t/\tau} \qquad (14.59)$$

where $E_0 = \frac{1}{2}kA^2$ is the initial energy at $t = 0$ and where we used $(z^m)^2 = z^{2m}$. In other words, **the oscillator's mechanical energy decays exponentially with time constant τ.**

As **FIGURE 14.26** shows, the time constant is the amount of time needed for the energy to decay to e^{-1}, or 37%, of its initial value. We say that the time constant τ measures the "characteristic time" during which the energy of the oscillation is dissipated. Roughly two-thirds of the initial energy is gone after one time constant has elapsed, and nearly 90% has dissipated after two time constants have gone by.

For practical purposes, we can speak of the time constant as the *lifetime* of the oscillation—about how long it lasts. Mathematically, there is never a time when the oscillation is "over." The decay approaches zero asymptotically, but it never gets there in any finite time. The best we can do is define a characteristic time when the motion is "almost over," and that is what the time constant τ does.

FIGURE 14.26 Exponential decay of the mechanical energy of an oscillator.



EXAMPLE 14.11 A damped pendulum

A 500 g mass swings on a 60-cm-string as a pendulum. The amplitude is observed to decay to half its initial value after 35.0 s.

a. What is the time constant for this oscillator?
b. At what time will the *energy* have decayed to half its initial value?

MODEL The motion is a damped oscillation.

SOLVE a. The initial amplitude at $t = 0$ is $x_{\text{max}} = A$. At $t = 35.0$ s the amplitude is $x_{\text{max}} = \frac{1}{2}A$. The amplitude of oscillation at time t is given by Equation 14.57:

$$x_{\text{max}}(t) = Ae^{-bt/2m} = Ae^{-t/2\tau}$$

In this case,

$$\frac{1}{2}A = Ae^{-(35.0\text{ s})/2\tau}$$

Notice that we do not need to know A itself because it cancels out. To solve for τ, we take the natural logarithm of both sides of the equation:

$$\ln\left(\frac{1}{2}\right) = -\ln 2 = \ln e^{-(35.0\text{ s})/2\tau} = -\frac{35.0\text{ s}}{2\tau}$$

This is easily rearranged to give

$$\tau = \frac{35.0\text{ s}}{2\ln 2} = 25.2\text{ s}$$

If desired, we could now determine the damping constant to be $b = m/\tau = 0.020$ kg/s.

b. The energy at time t is given by

$$E(t) = E_0 e^{-t/\tau}$$

The time at which an exponential decay is reduced to $\frac{1}{2}E_0$, half its initial value, has a special name. It is called the **half-life** and given the symbol $t_{1/2}$. The concept of the half-life is widely used in applications such as radioactive decay. To relate $t_{1/2}$ to τ, we first write

$$E(\text{at } t = t_{1/2}) = \frac{1}{2}E_0 = E_0 e^{-t_{1/2}/\tau}$$

The E_0 cancels, giving

$$\frac{1}{2} = e^{-t_{1/2}/\tau}$$

Again, we take the natural logarithm of both sides:

$$\ln\left(\frac{1}{2}\right) = -\ln 2 = \ln e^{-t_{1/2}/\tau} = -t_{1/2}/\tau$$

Finally, we solve for $t_{1/2}$:

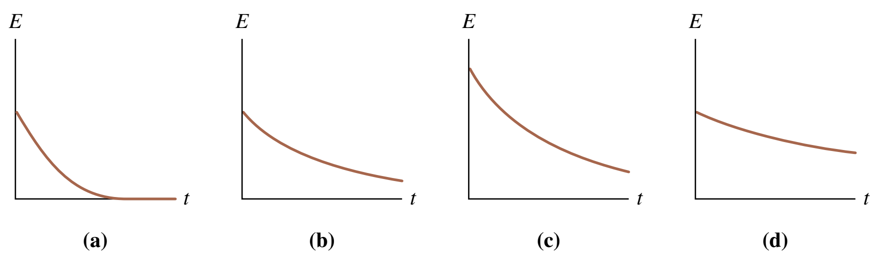
$$t_{1/2} = \tau \ln 2 = 0.693\tau$$

This result that $t_{1/2}$ is 69% of τ is valid for any exponential decay. In this particular problem, half the energy is gone at

$$t_{1/2} = (0.693)(25.2\text{ s}) = 17.5\text{ s}$$

ASSESS The oscillator loses energy faster than it loses amplitude. This is what we should expect because the energy depends on the *square* of the amplitude.

STOP TO THINK 14.6 Rank in order, from largest to smallest, the time constants τ_a to τ_d of the decays shown in the figure. All the graphs have the same scale.



14.8 Driven Oscillations and Resonance

Thus far we have focused on the free oscillations of an isolated system. Some initial disturbance displaces the system from equilibrium, and it then oscillates freely until its energy is dissipated. These are very important situations, but they do not exhaust the possibilities. Another important situation is an oscillator that is subjected to a periodic external force. Its motion is called a **driven oscillation.**

A simple example of a driven oscillation is pushing a child on a swing, where your push is a periodic external force applied to the swing. A more complex example is a car driving over a series of equally spaced bumps. Each bump causes a periodic upward force on the car's shock absorbers, which are big, heavily damped springs. The electromagnetic coil on the back of a loudspeaker cone provides a periodic magnetic force to drive the cone back and forth, causing it to send out sound waves. Air turbulence moving across the wings of an aircraft can exert periodic forces on the wings and other aerodynamic surfaces, causing them to vibrate if they are not properly designed.

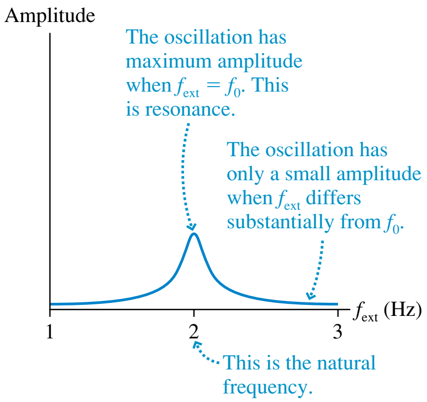
As these examples suggest, driven oscillations have many important applications. However, driven oscillations are a mathematically complex subject. We will simply hint at some of the results, saving the details for more advanced classes.

Consider an oscillating system that, when left to itself, oscillates at a frequency f_0. We will call this the **natural frequency** of the oscillator. The natural frequency for a mass on a spring is $\sqrt{k/m}/2\pi$, but it might be given by some other expression for another type of oscillator. Regardless of the expression, f_0 is simply the frequency of the system if it is displaced from equilibrium and released.

Suppose that this system is subjected to a *periodic* external force of frequency f_{ext}. This frequency, which is called the **driving frequency,** is completely independent of the oscillator's natural frequency f_0. Somebody or something in the environment selects the frequency f_{ext} of the external force, causing the force to push on the system f_{ext} times every second.

Although it is possible to solve Newton's second law with an external driving force, we will be content to look at a graphical representation of the solution. The most important result is that the oscillation amplitude depends very sensitively on the frequency f_{ext} of the driving force. The response to the driving frequency is shown in **FIGURE 14.27** for a system with $m = 1.0$ kg, a natural frequency $f_0 = 2.0$ Hz, and a damping constant $b = 0.20$ kg/s. This graph of amplitude versus driving frequency, called the **response curve,** occurs in many different applications.

FIGURE 14.27 The response curve shows the amplitude of a driven oscillator at frequencies near its natural frequency of 2.0 Hz.

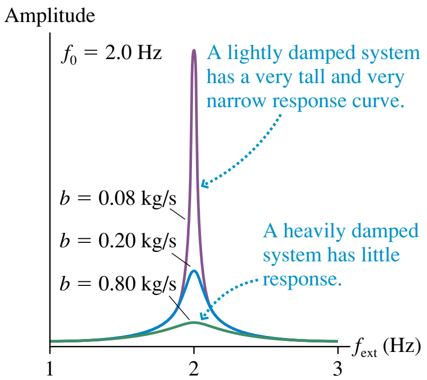


When the driving frequency is substantially different from the oscillator's natural frequency, at the right and left edges of Figure 14.27, the system oscillates but the amplitude is very small. The system simply does not respond well to a driving frequency that differs much from f_0. As the driving frequency gets closer and closer to the natural frequency, the amplitude of the oscillation rises dramatically. After all, f_0 is the frequency at which the system "wants" to oscillate, so it is quite happy to respond to a driving frequency near f_0. Hence the amplitude reaches a maximum when the driving frequency exactly matches the system's natural frequency: $f_{ext} = f_0$.

The amplitude can become exceedingly large when the frequencies match, especially if the damping constant is very small. **FIGURE 14.28** shows the same oscillator with three different values of the damping constant. There's very little response if the damping constant is increased to 0.80 kg/s, but the amplitude for $f_{ext} = f_0$ becomes very large when the damping constant is reduced to 0.08 kg/s. This large-amplitude response to a driving force whose frequency matches the natural frequency of the system is a phenomenon called **resonance.** The condition for resonance is

$$f_{ext} = f_0 \quad \text{(resonance condition)} \tag{14.60}$$

FIGURE 14.28 The resonance amplitude becomes higher and narrower as the damping constant decreases.



Within the context of driven oscillations, the natural frequency f_0 is often called the **resonance frequency.**

An important feature of Figure 14.28 is how the amplitude and width of the resonance depend on the damping constant. A heavily damped system responds fairly little, even at resonance, but it responds to a wide range of driving frequencies. Very lightly damped systems can reach exceptionally high amplitudes, but notice that the range of frequencies to which the system responds becomes narrower and narrower as b decreases.

This allows us to understand why a few singers can break crystal goblets but not inexpensive, everyday glasses. An inexpensive glass gives a "thud" when tapped, but a fine crystal goblet "rings" for several seconds. In physics terms, the goblet has a much longer time constant than the glass. That, in turn, implies that the goblet is very lightly damped while the ordinary glass is heavily damped (because the internal forces within the glass are not those of a high-quality crystal structure).

The singer causes a sound wave to impinge on the goblet, exerting a small driving force at the frequency of the note she is singing. If the singer's frequency matches the natural frequency of the goblet—resonance! Only the lightly damped goblet, like the top curve in Figure 14.28, can reach amplitudes large enough to shatter. The restriction, though, is that its natural frequency has to be matched very precisely. The sound also has to be very loud.

A singer or musical instrument can shatter a crystal goblet by matching the goblet's natural oscillation frequency.

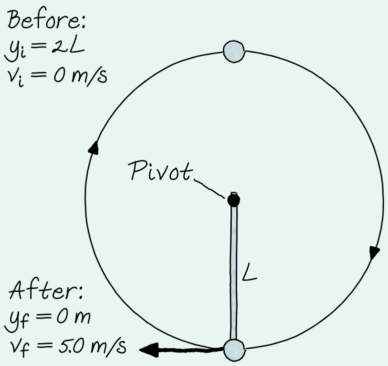
CHALLENGE EXAMPLE 14.12 A swinging pendulum

A pendulum consists of a massless, rigid rod with a mass at one end. The other end is pivoted on a frictionless pivot so that the rod can rotate in a complete circle. The pendulum is inverted, with the mass directly above the pivot point, then released. The speed of the mass as it passes through the lowest point is 5.0 m/s. If the pendulum later undergoes small-amplitude oscillations at the bottom of the arc, what will its frequency be?

MODEL This is a simple pendulum because the rod is massless. However, our analysis of a pendulum used the small-angle approximation. It applies only to the small-amplitude oscillations at the end, *not* to the pendulum swinging down from the inverted position. Fortunately, energy is conserved throughout, so we can analyze the big swing using conservation of mechanical energy.

VISUALIZE **FIGURE 14.29** is a pictorial representation of the pendulum swinging down from the inverted position. The pendulum length is L, so the initial height is $2L$.

FIGURE 14.29 Before-and-after pictorial representation of the pendulum swinging down from an inverted position.



SOLVE The frequency of a simple pendulum is $f = \sqrt{g/L}/2\pi$. We're not given L, but we can find it by analyzing the pendulum's swing down from an inverted position. Mechanical energy is conserved, and the only potential energy is gravitational potential energy. Conservation of mechanical energy $K_f + U_{gf} = K_i + U_{gi}$, with $U_g = mgy$, is

$$\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i$$

The mass cancels, which is good since we don't know it, and two terms are zero. Thus

$$\frac{1}{2}v_f^2 = g(2L) = 2gL$$

Solving for L, we find

$$L = \frac{v_f^2}{4g} = \frac{(5.0 \text{ m/s})^2}{4(9.80 \text{ m/s}^2)} = 0.638 \text{ m}$$

Now we can calculate the frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.80 \text{ m/s}^2}{0.638 \text{ m}}} = 0.62 \text{ Hz}$$

ASSESS The frequency corresponds to a period of about 1.5 s, which seems reasonable.

A consistent 4-step approach provides a problem-solving framework throughout the book (and all supplements): students learn the importance of making assumptions (in the MODEL step), gathering information, and making sketches (in the VISUALIZE step) before treating the problem mathematically (SOLVE) and then analyzing their result (ASSESS).

Challenge Examples illustrate how to integrate multiple concepts and use more sophisticated reasoning in problem-solving, ensuring an optimal range of worked examples for students to study in preparation for homework problems.

Unique and critically acclaimed **visual chapter summaries** consolidate understanding by providing each concept in words, math, and figures and organizing these into a vertical hierarchy—from General Principles (top) to Applications (bottom).

SUMMARY

The goal of Chapter 14 has been to understand systems that oscillate with simple harmonic motion.

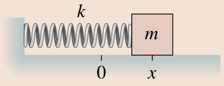
General Principles

Dynamics

SHM occurs when a linear restoring force acts to return a system to an equilibrium position.

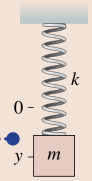
Horizontal spring

$(F_{net})_x = -kx$



Vertical spring
The origin is at the equilibrium position $\Delta L = mg/k$.

$(F_{net})_y = -ky$

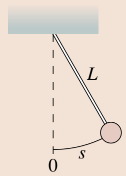


Both: $\omega = \sqrt{\dfrac{k}{m}}$ $\quad T = 2\pi\sqrt{\dfrac{m}{k}}$

Pendulum

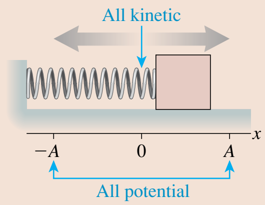
$(F_{net})_t = -\left(\dfrac{mg}{L}\right)s$

$\omega = \sqrt{\dfrac{g}{L}}$ $\quad T = 2\pi\sqrt{\dfrac{L}{g}}$



Energy

If there is **no friction** or dissipation, kinetic and potential energy are alternately transformed into each other, but the total mechanical energy $E = K + U$ is conserved.

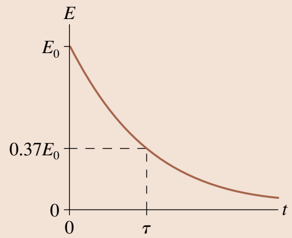


$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$
$$= \frac{1}{2}m(v_{max})^2$$
$$= \frac{1}{2}kA^2$$

In a **damped system,** the energy decays exponentially

$$E = E_0 e^{-t/\tau}$$

where τ is the time constant.



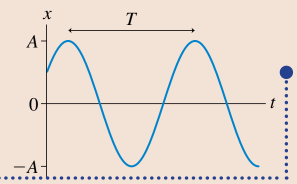
Important Concepts

Simple harmonic motion (SHM) is a sinusoidal oscillation with period T and amplitude A.

Frequency $f = \dfrac{1}{T}$

Angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T}$$



Position $x(t) = A\cos(\omega t + \phi_0)$

$$= A\cos\left(\frac{2\pi t}{T} + \phi_0\right)$$

Velocity $v_x(t) = -v_{max}\sin(\omega t + \phi_0)$ with maximum speed $v_{max} = \omega A$

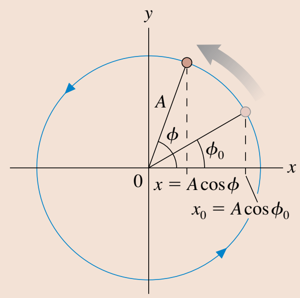
Acceleration $a_x(t) = -\omega^2 x(t) = -\omega^2 A\cos(\omega t + \phi_0)$

SHM is the projection onto the x-axis of **uniform circular motion.**

$\phi = \omega t + \phi_0$ is the phase

The position at time t is

$$x(t) = A\cos\phi$$
$$= A\cos(\omega t + \phi_0)$$



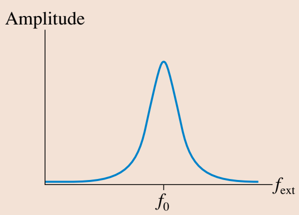
The phase constant ϕ_0 determines the initial conditions:

$x_0 = A\cos\phi_0 \qquad v_{0x} = -\omega A\sin\phi_0$

Applications

Resonance

When a system is driven by a periodic external force, it responds with a large-amplitude oscillation if $f_{ext} \approx f_0$, where f_0 is the system's natural oscillation frequency, or **resonant frequency.**

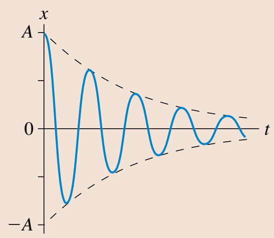


Damping

If there is a drag force $\vec{D} = -b\vec{v}$, where b is the damping constant, then (for lightly damped systems)

$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0)$$

The time constant for energy loss is $\tau = m/b$.



Terms and Notation

oscillatory motion	amplitude, A	linear restoring force	natural frequency, f_0
oscillator	angular frequency, ω	damped oscillation	driving frequency, f_{ext}
period, T	phase, ϕ	damping constant, b	response curve
frequency, f	phase constant, ϕ_0	envelope	resonance
hertz, Hz	restoring force	time constant, τ	resonance frequency, f_0
simple harmonic motion, SHM	equation of motion	half-life, $t_{1/2}$	
	small-angle approximation	driven oscillation	

CONCEPTUAL QUESTIONS

Conceptual Questions require careful reasoning and can be used for group discussions or individual work.

1. A block oscillating on a spring has period $T = 2$ s. What is the period if:
   a. The block's mass is doubled? Explain. Note that you do not know the value of either m or k, so do *not* assume any particular values for them. The required analysis involves thinking about ratios.
   b. The value of the spring constant is quadrupled?
   c. The oscillation amplitude is doubled while m and k are unchanged?
2. A pendulum on Planet X, where the value of g is unknown, oscillates with a period $T = 2$ s. What is the period of this pendulum if:
   a. Its mass is doubled? Explain. Note that you do not know the value of m, L, or g, so do not assume any specific values. The required analysis involves thinking about ratios.
   b. Its length is doubled?
   c. Its oscillation amplitude is doubled?
3. FIGURE Q14.3 shows a position-versus-time graph for a particle in SHM. What are (a) the amplitude A, (b) the angular frequency ω, and (c) the phase constant ϕ_0? Explain.

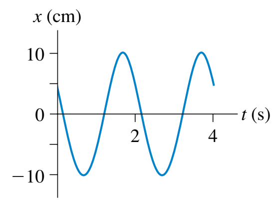


FIGURE Q14.3

4. Equation 14.25 states that $\frac{1}{2}kA^2 = \frac{1}{2}m(v_{max})^2$. What does this mean? Write a couple of sentences explaining how to interpret this equation.
5. A block oscillating on a spring has an amplitude of 20 cm. What will the amplitude be if the total energy is doubled? Explain.
6. A block oscillating on a spring has a maximum speed of 20 cm/s. What will the block's maximum speed be if the total energy is doubled? Explain.
7. FIGURE Q14.7 shows a position-versus-time graph for a particle in SHM.
   a. What is the phase constant ϕ_0? Explain.
   b. What is the phase of the particle at each of the three numbered points on the graph?

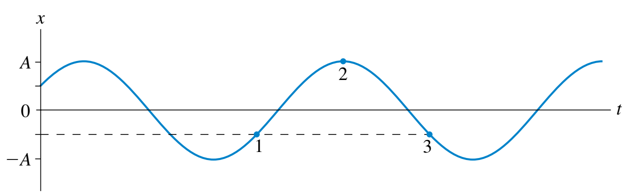


FIGURE Q14.7

8. FIGURE Q14.8 shows a velocity-versus-time graph for a particle in SHM.
   a. What is the phase constant ϕ_0? Explain.
   b. What is the phase of the particle at each of the three numbered points on the graph?

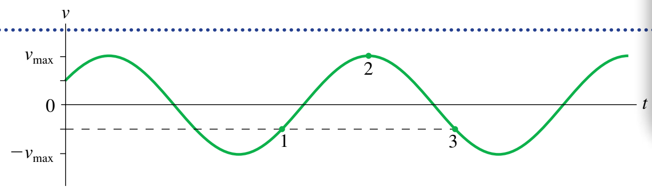


FIGURE Q14.8

9. FIGURE Q14.9 shows the potential-energy diagram and the total energy line of a particle oscillating on a spring.
   a. What is the spring's equilibrium length?
   b. Where are the turning points of the motion? Explain.
   c. What is the particle's maximum kinetic energy?
   d. What will be the turning points if the particle's total energy is doubled?

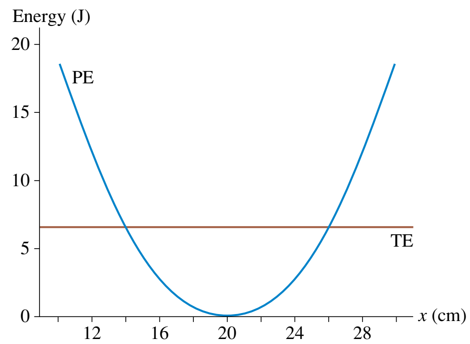


FIGURE Q14.9

10. Suppose the damping constant b of an oscillator increases.
    a. Is the medium more resistive or less resistive?
    b. Do the oscillations damp out more quickly or less quickly?
    c. Is the time constant τ increased or decreased?
11. a. Describe the difference between τ and T. Don't just *name* them; say what is different about the physical concepts they represent.
    b. Describe the difference between τ and $t_{1/2}$.
12. What is the difference between the driving frequency and the natural frequency of an oscillator?

The end-of-chapter problems are rated by students to show difficulty level with the variety expanded to include more real-world, challenging, and explicitly calculus-based problems.

Exercises (for each section) allow students to build their skills and confidence with straightforward, one-step questions.

EXERCISES AND PROBLEMS

Problems labeled [shaded] integrate material from earlier chapters.

Exercises

Section 14.1 Simple Harmonic Motion

1. | When a guitar string plays the note "A," the string vibrates at 440 Hz. What is the period of the vibration?
2. | An air-track glider attached to a spring oscillates between the 10 cm mark and the 60 cm mark on the track. The glider completes 10 oscillations in 33 s. What are the (a) period, (b) frequency, (c) angular frequency, (d) amplitude, and (e) maximum speed of the glider?
3. || An air-track glider is attached to a spring. The glider is pulled to the right and released from rest at $t = 0$ s. It then oscillates with a period of 2.0 s and a maximum speed of 40 cm/s.
   a. What is the amplitude of the oscillation?
   b. What is the glider's position at $t = 0.25$ s?

Section 14.2 Simple Harmonic Motion and Circular Motion

4. | What are the (a) amplitude, (b) frequency, and (c) phase constant of the oscillation shown in FIGURE EX14.4?

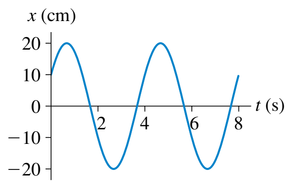


FIGURE EX14.4

5. || What are the (a) amplitude, (b) frequency, and (c) phase constant of the oscillation shown in FIGURE EX14.5?

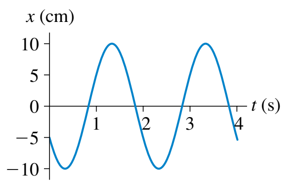


FIGURE EX14.5

6. || An object in simple harmonic motion has an amplitude of 4.0 cm, a frequency of 2.0 Hz, and a phase constant of $2\pi/3$ rad. Draw a position graph showing two cycles of the motion.
7. || An object in simple harmonic motion has an amplitude of 8.0 cm, a frequency of 0.25 Hz, and a phase constant of $-\pi/2$ rad. Draw a position graph showing two cycles of the motion.
8. | An object in simple harmonic motion has amplitude 4.0 cm and frequency 4.0 Hz, and at $t = 0$ s it passes through the equilibrium point moving to the right. Write the function $x(t)$ that describes the object's position.
9. | An object in simple harmonic motion has amplitude 8.0 cm and frequency 0.50 Hz. At $t = 0$ s it has its most negative position. Write the function $x(t)$ that describes the object's position.
10. || An air-track glider attached to a spring oscillates with a period of 1.5 s. At $t = 0$ s the glider is 5.00 cm left of the equilibrium position and moving to the right at 36.3 cm/s.
    a. What is the phase constant?
    b. What is the phase at $t = 0$ s, 0.5 s, 1.0 s, and 1.5 s?

Section 14.3 Energy in Simple Harmonic Motion

Section 14.4 The Dynamics of Simple Harmonic Motion

11. | A block attached to a spring with unknown spring constant oscillates with a period of 2.0 s. What is the period if
    a. The mass is doubled?
    b. The mass is halved?
    c. The amplitude is doubled?
    d. The spring constant is doubled?
    Parts a to d are independent questions, each referring to the initial situation.
12. || A 200 g air-track glider is attached to a spring. The glider is pushed in 10 cm and released. A student with a stopwatch finds that 10 oscillations take 12.0 s. What is the spring constant?
13. || A 200 g mass attached to a horizontal spring oscillates at a frequency of 2.0 Hz. At $t = 0$ s, the mass is at $x = 5.0$ cm and has $v_x = -30$ cm/s. Determine:
    a. The period.
    b. The angular frequency.
    c. The amplitude.
    d. The phase constant.
    e. The maximum speed.
    f. The maximum acceleration.
    g. The total energy.
    h. The position at $t = 0.40$ s.
14. | The position of a 50 g oscillating mass is given by $x(t) = (2.0\text{ cm})\cos(10t - \pi/4)$, where t is in s. Determine:
    a. The amplitude.
    b. The period.
    c. The spring constant.
    d. The phase constant.
    e. The initial conditions.
    f. The maximum speed.
    g. The total energy.
    h. The velocity at $t = 0.40$ s.
15. || A 1.0 kg block is attached to a spring with spring constant 16 N/m. While the block is sitting at rest, a student hits it with a hammer and almost instantaneously gives it a speed of 40 cm/s. What are
    a. The amplitude of the subsequent oscillations?
    b. The block's speed at the point where $x = \frac{1}{2}A$?

Section 14.5 Vertical Oscillations

16. | A spring is hanging from the ceiling. Attaching a 500 g physics book to the spring causes it to stretch 20 cm in order to come to equilibrium.
    a. What is the spring constant?
    b. From equilibrium, the book is pulled down 10 cm and released. What is the period of oscillation?
    c. What is the book's maximum speed?
17. || A spring with spring constant 15 N/m hangs from the ceiling. A ball is attached to the spring and allowed to come to rest. It is then pulled down 6.0 cm and released. If the ball makes 30 oscillations in 20 s, what are its (a) mass and (b) maximum speed?
18. || A spring is hung from the ceiling. When a block is attached to its end, it stretches 2.0 cm before reaching its new equilibrium length. The block is then pulled down slightly and released. What is the frequency of oscillation?

Section 14.6 The Pendulum

19. | A mass on a string of unknown length oscillates as a pendulum with a period of 4.0 s. What is the period if
    a. The mass is doubled?

b. The string length is doubled?
c. The string length is halved?
d. The amplitude is doubled?

Parts a to d are independent questions, each referring to the initial situation.

20. A 200 g ball is tied to a string. It is pulled to an angle of 8.0° and released to swing as a pendulum. A student with a stopwatch finds that 10 oscillations take 12 s. How long is the string?

21. What is the period of a 1.0-m-long pendulum on (a) the earth and (b) Venus?

22. What is the length of a pendulum whose period on the moon matches the period of a 2.0-m-long pendulum on the earth?

23. Astronauts on the first trip to Mars take along a pendulum that has a period on earth of 1.50 s. The period on Mars turns out to be 2.45 s. What is the free-fall acceleration on Mars?

24. A uniform steel bar swings from a pivot at one end with a period of 1.2 s. How long is the bar?

Section 14.7 Damped Oscillations

Section 14.8 Driven Oscillations and Resonance

25. A 2.0 g spider is dangling at the end of a silk thread. You can make the spider bounce up and down on the thread by tapping lightly on his feet with a pencil. You soon discover that you can give the spider the largest amplitude on his little bungee cord if you tap exactly once every second. What is the spring constant of the silk thread?

26. The amplitude of an oscillator decreases to 36.8% of its initial value in 10.0 s. What is the value of the time constant?

27. Sketch a position graph from $t = 0$ s to $t = 10$ s of a damped oscillator having a frequency of 1.0 Hz and a time constant of 4.0 s.

28. In a science museum, a 110 kg brass pendulum bob swings at the end of a 15.0-m-long wire. The pendulum is started at exactly 8:00 A.M. every morning by pulling it 1.5 m to the side and releasing it. Because of its compact shape and smooth surface, the pendulum's damping constant is only 0.010 kg/s. At exactly 12:00 noon, how many oscillations will the pendulum have completed and what is its amplitude?

29. BIO Vision is blurred if the head is vibrated at 29 Hz because the vibrations are resonant with the natural frequency of the eyeball in its socket. If the mass of the eyeball is 7.5 g, a typical value, what is the effective spring constant of the musculature that holds the eyeball in the socket?

Problems

30. FIGURE P14.30 is the velocity-versus-time graph of a particle in simple harmonic motion.
a. What is the amplitude of the oscillation?
b. What is the phase constant?
c. What is the position at $t = 0$ s?

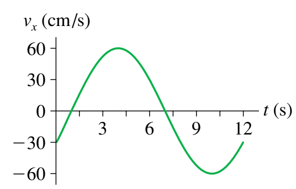


FIGURE P14.30

31. FIGURE P14.31 is the position-versus-time graph of a particle in simple harmonic motion.
a. What is the phase constant?
b. What is the velocity at $t = 0$ s?
c. What is v_{max}?

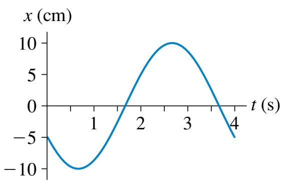


FIGURE P14.31

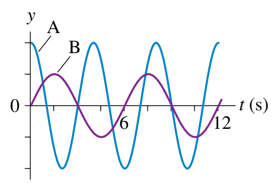


FIGURE P14.32

32. The two graphs in FIGURE P14.32 are for two different vertical mass-spring systems. If both systems have the same mass, what is the ratio k_A/k_B of their spring constants?

33. An object in SHM oscillates with a period of 4.0 s and an amplitude of 10 cm. How long does the object take to move from $x = 0.0$ cm to $x = 6.0$ cm?

34. A 1.0 kg block oscillates on a spring with spring constant 20 N/m. At $t = 0$ s the block is 20 cm to the right of the equilibrium position and moving to the left at a speed of 100 cm/s. Determine (a) the period and (b) the amplitude.

35. BIO Astronauts in space cannot weigh themselves by standing on a bathroom scale. Instead, they determine their mass by oscillating on a large spring. Suppose an astronaut attaches one end of a large spring to her belt and the other end to a hook on the wall of the space capsule. A fellow astronaut then pulls her away from the wall and releases her. The spring's length as a function of time is shown in FIGURE P14.35.
a. What is her mass if the spring constant is 240 N/m?
b. What is her speed when the spring's length is 1.2 m?

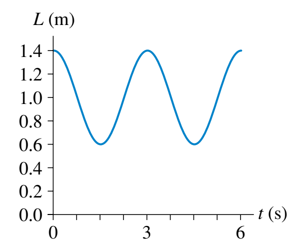


FIGURE P14.35

36. The motion of a particle is given by $x(t) = (25\text{ cm})\cos(10t)$, where t is in s. At what time is the kinetic energy twice the potential energy?

37. a. When the displacement of a mass on a spring is $\frac{1}{2}A$, what fraction of the energy is kinetic energy and what fraction is potential energy?
b. At what displacement, as a fraction of A, is the energy half kinetic and half potential?

38. For a particle in simple harmonic motion, show that $v_{max} = (\pi/2)v_{avg}$ where v_{avg} is the average speed during one cycle of the motion.

39. A 100 g block attached to a spring with spring constant 2.5 N/m oscillates horizontally on a frictionless table. Its velocity is 20 cm/s when $x = -5.0$ cm.
a. What is the amplitude of oscillation?
b. What is the block's maximum acceleration?
c. What is the block's position when the acceleration is maximum?
d. What is the speed of the block when $x = 3.0$ cm?

Problems (spanning concepts from the whole chapter), require in-depth reasoning and planning, and allow students to practice their problem-solving strategies. Context-rich problems require students to simplify and model more complex real-world situations. Specifically labeled problems integrate concepts from multiple previous chapters.

Bio problems are set in life-science, bioengineering, or biomedical contexts.

Data-based problems allow students to practice drawing conclusions from data (as demonstrated in the new data-based examples in the text).

40. A block on a spring is pulled to the right and released at $t = 0$ s. It passes $x = 3.00$ cm at $t = 0.685$ s, and it passes $x = -3.00$ cm at $t = 0.886$ s.
 a. What is the angular frequency?
 b. What is the amplitude?
 Hint: $\cos(\pi - \theta) = -\cos\theta$.

41. A 300 g oscillator has a speed of 95.4 cm/s when its displacement is 3.0 cm and 71.4 cm/s when its displacement is 6.0 cm. What is the oscillator's maximum speed?

42. BIO An ultrasonic transducer, of the type used in medical ultrasound imaging, is a very thin disk ($m = 0.10$ g) driven back and forth in SHM at 1.0 MHz by an electromagnetic coil.
 a. The maximum restoring force that can be applied to the disk without breaking it is 40,000 N. What is the maximum oscillation amplitude that won't rupture the disk?
 b. What is the disk's maximum speed at this amplitude?

43. A 5.0 kg block hangs from a spring with spring constant 2000 N/m. The block is pulled down 5.0 cm from the equilibrium position and given an initial velocity of 1.0 m/s back toward equilibrium. What are the (a) frequency, (b) amplitude, and (c) total mechanical energy of the motion?

44. Your lab instructor has asked you to measure a spring constant using a dynamic method—letting it oscillate—rather than a static method of stretching it. You and your lab partner suspend the spring from a hook, hang different masses on the lower end, and start them oscillating. One of you uses a meter stick to measure the amplitude, the other uses a stopwatch to time 10 oscillations. Your data are as follows:

Mass (g)	Amplitude (cm)	Time (s)
100	6.5	7.8
150	5.5	9.8
200	6.0	10.9
250	3.5	12.4

Use the best-fit line of an appropriate graph to determine the spring constant.

45. A 200 g block hangs from a spring with spring constant 10 N/m. At $t = 0$ s the block is 20 cm below the equilibrium point and moving upward with a speed of 100 cm/s. What are the block's
 a. Oscillation frequency?
 b. Distance from equilibrium when the speed is 50 cm/s?
 c. Distance from equilibrium at $t = 1.0$ s?

46. A spring with spring constant k is suspended vertically from a support and a mass m is attached. The mass is held at the point where the spring is not stretched. Then the mass is released and begins to oscillate. The lowest point in the oscillation is 20 cm below the point where the mass was released. What is the oscillation frequency?

47. While grocery shopping, you put several apples in the spring scale in the produce department. The scale reads 20 N, and you use your ruler (which you always carry with you) to discover that the pan goes down 9.0 cm when the apples are added. If you tap the bottom of the apple-filled pan to make it bounce up and down a little, what is its oscillation frequency? Ignore the mass of the pan.

48. A compact car has a mass of 1200 kg. Assume that the car has one spring on each wheel, that the springs are identical, and that the mass is equally distributed over the four springs.
 a. What is the spring constant of each spring if the empty car bounces up and down 2.0 times each second?
 b. What will be the car's oscillation frequency while carrying four 70 kg passengers?

49. The two blocks in FIGURE P14.49 oscillate on a frictionless surface with a period of 1.5 s. The upper block just begins to slip when the amplitude is increased to 40 cm. What is the coefficient of static friction between the two blocks?

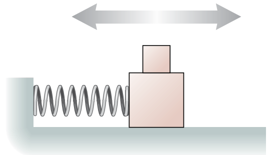


FIGURE P14.49

50. BIO It has recently become possible to "weigh" DNA molecules by measuring the influence of their mass on a nano-oscillator. FIGURE P14.50 shows a thin rectangular cantilever etched out of silicon (density 2300 kg/m^3) with a small gold dot at the end. If pulled down and released, the end of the cantilever vibrates with simple harmonic motion, moving up and down like a diving board after a jump. When bathed with DNA molecules whose ends have been modified to bind with gold, one or more molecules may attach to the gold dot. The addition of their mass causes a very slight—but measurable—decrease in the oscillation frequency.

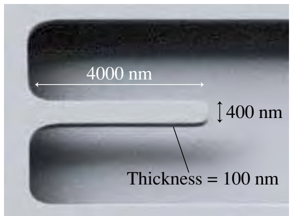


FIGURE P14.50

A vibrating cantilever of mass M can be modeled as a block of mass $\frac{1}{3}M$ attached to a spring. (The factor of $\frac{1}{3}$ arises from the moment of inertia of a bar pivoted at one end.) Neither the mass nor the spring constant can be determined very accurately— perhaps to only two significant figures—but the oscillation frequency can be measured with very high precision simply by counting the oscillations. In one experiment, the cantilever was initially vibrating at exactly 12 MHz. Attachment of a DNA molecule caused the frequency to decrease by 50 Hz. What was the mass of the DNA?

51. It is said that Galileo discovered a basic principle of the pendulum—that the period is independent of the amplitude—by using his pulse to time the period of swinging lamps in the cathedral as they swayed in the breeze. Suppose that one oscillation of a swinging lamp takes 5.5 s.
 a. How long is the lamp chain?
 b. What maximum speed does the lamp have if its maximum angle from vertical is 3.0°?

52. A 100 g mass on a 1.0-m-long string is pulled 8.0° to one side and released. How long does it take for the pendulum to reach 4.0° on the opposite side?

53. BIO Orangutans can move by brachiation, swinging like a pendulum beneath successive handholds. If an orangutan has arms that are 0.90 m long and repeatedly swings to a 20° angle, taking one swing after another, estimate its speed of forward motion in m/s. While this is somewhat beyond the range of validity of the small-angle approximation, the standard results for a pendulum are adequate for making an estimate.

54. Show that Equation 14.51 for the angular frequency of a physical pendulum gives Equation 14.48 when applied to a simple pendulum of a mass on a string.

55. A 15-cm-long, 200 g rod is pivoted at one end. A 20 g ball of clay is stuck on the other end. What is the period if the rod and clay swing as a pendulum?

56. A uniform rod of mass M and length L swings as a pendulum on a pivot at distance $L/4$ from one end of the rod. Find an expression for the frequency f of small-angle oscillations.

57. A solid sphere of mass M and radius R is suspended from a thin rod, as shown in FIGURE P14.57. The sphere can swing back and forth at the bottom of the rod. Find an expression for the frequency f of small-angle oscillations.

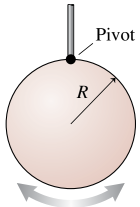


FIGURE P14.57

58. A geologist needs to determine the local value of g. Unfortunately, his only tools are a meter stick, a saw, and a stopwatch. He starts by hanging the meter stick from one end and measuring its frequency as it swings. He then saws off 20 cm—using the centimeter markings—and measures the frequency again. After two more cuts, these are his data:

Length (cm)	Frequency (Hz)
100	0.61
80	0.67
60	0.79
40	0.96

Use the best-fit line of an appropriate graph to determine the local value of g.

59. BIO Interestingly, there have been several studies using cadavers to determine the moments of inertia of human body parts, information that is important in biomechanics. In one study, the center of mass of a 5.0 kg lower leg was found to be 18 cm from the knee. When the leg was allowed to pivot at the knee and swing freely as a pendulum, the oscillation frequency was 1.6 Hz. What was the moment of inertia of the lower leg about the knee joint?

60. A 500 g air-track glider attached to a spring with spring constant 10 N/m is sitting at rest on a frictionless air track. A 250 g glider is pushed toward it from the far end of the track at a speed of 120 cm/s. It collides with and sticks to the 500 g glider. What are the amplitude and period of the subsequent oscillations?

61. A 200 g block attached to a horizontal spring is oscillating with an amplitude of 2.0 cm and a frequency of 2.0 Hz. Just as it passes through the equilibrium point, moving to the right, a sharp blow directed to the left exerts a 20 N force for 1.0 ms. What are the new (a) frequency and (b) amplitude?

62. FIGURE P14.62 is a top view of an object of mass m connected between two stretched rubber bands of length L. The object rests on a frictionless surface. At equilibrium, the tension in each rubber band is T. Find an expression for the frequency of oscillations *perpendicular* to the rubber bands. Assume the amplitude is sufficiently small that the magnitude of the tension in the rubber bands is essentially unchanged as the mass oscillates.

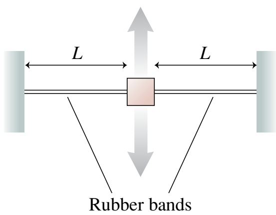


FIGURE P14.62

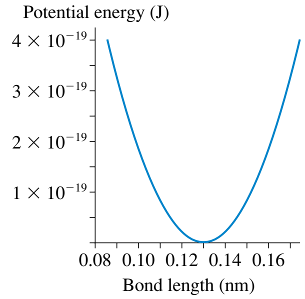


FIGURE P14.63

An Increased emphasis on symbolic answers encourages students to work algebraically. The Student Workbook also contains new exercises to help students work through symbolic solutions.

63. A molecular bond can be modeled as a spring between two atoms that vibrate with simple harmonic motion. FIGURE P14.63 shows an SHM approximation for the potential energy of an HCl molecule. For $E < 4 \times 10^{-19}$ J it is a good approximation to the more accurate HCl potential-energy curve that was shown in Figure 10.31. Because the chlorine atom is so much more massive than the hydrogen atom, it is reasonable to assume that the hydrogen atom ($m = 1.67 \times 10^{-27}$ kg) vibrates back and forth while the chlorine atom remains at rest. Use the graph to estimate the vibrational frequency of the HCl molecule.

64. An ice cube can slide around the inside of a vertical circular hoop of radius R. It undergoes small-amplitude oscillations if displaced slightly from the equilibrium position at the lowest point. Find an expression for the period of these small-amplitude oscillations.

65. A penny rides on top of a piston as it undergoes vertical simple harmonic motion with an amplitude of 4.0 cm. If the frequency is low, the penny rides up and down without difficulty. If the frequency is steadily increased, there comes a point at which the penny leaves the surface.
   a. At what point in the cycle does the penny first lose contact with the piston?
   b. What is the maximum frequency for which the penny just barely remains in place for the full cycle?

66. On your first trip to Planet X you happen to take along a 200 g mass, a 40-cm-long spring, a meter stick, and a stopwatch. You're curious about the free-fall acceleration on Planet X, where ordinary tasks seem easier than on earth, but you can't find this information in your Visitor's Guide. One night you suspend the spring from the ceiling in your room and hang the mass from it. You find that the mass stretches the spring by 31.2 cm. You then pull the mass down 10.0 cm and release it. With the stopwatch you find that 10 oscillations take 14.5 s. Based on this information, what is g?

67. The 15 g head of a bobble-head doll oscillates in SHM at a frequency of 4.0 Hz.
   a. What is the spring constant of the spring on which the head is mounted?
   b. The amplitude of the head's oscillations decreases to 0.5 cm in 4.0 s. What is the head's damping constant?

68. An oscillator with a mass of 500 g and a period of 0.50 s has an amplitude that decreases by 2.0% during each complete oscillation. If the initial amplitude is 10 cm, what will be the amplitude after 25 oscillations?

69. A spring with spring constant 15.0 N/m hangs from the ceiling. A 500 g ball is attached to the spring and allowed to come to rest. It is then pulled down 6.0 cm and released. What is the time constant if the ball's amplitude has decreased to 3.0 cm after 30 oscillations?

70. ||| A 250 g air-track glider is attached to a spring with spring constant 4.0 N/m. The damping constant due to air resistance is 0.015 kg/s. The glider is pulled out 20 cm from equilibrium and released. How many oscillations will it make during the time in which the amplitude decays to e^{-1} of its initial value?

71. || A 200 g oscillator in a vacuum chamber has a frequency of 2.0 Hz. When air is admitted, the oscillation decreases to 60% of its initial amplitude in 50 s. How many oscillations will have been completed when the amplitude is 30% of its initial value?

72. || Prove that the expression for $x(t)$ in Equation 14.55 is a solution to the equation of motion for a damped oscillator, Equation 14.54, if and only if the angular frequency ω is given by the expression in Equation 14.56.

73. || A block on a frictionless table is connected as shown in FIGURE P14.73 to two springs having spring constants k_1 and k_2. Show that the block's oscillation frequency is given by

$$f = \sqrt{f_1^2 + f_2^2}$$

where f_1 and f_2 are the frequencies at which it would oscillate if attached to spring 1 or spring 2 alone.

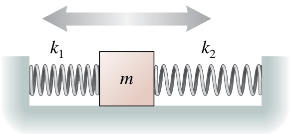


FIGURE P14.73

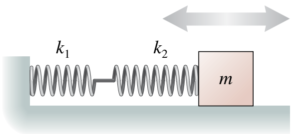


FIGURE P14.74

74. || A block on a frictionless table is connected as shown in FIGURE P14.74 to two springs having spring constants k_1 and k_2. Find an expression for the block's oscillation frequency f in terms of the frequencies f_1 and f_2 at which it would oscillate if attached to spring 1 or spring 2 alone.

Challenge Problems

75. A block hangs in equilibrium from a vertical spring. When a second identical block is added, the original block sags by 5.0 cm. What is the oscillation frequency of the two-block system?

76. A 1.00 kg block is attached to a horizontal spring with spring constant 2500 N/m. The block is at rest on a frictionless surface. A 10 g bullet is fired into the block, in the face opposite the spring, and sticks. What was the bullet's speed if the subsequent oscillations have an amplitude of 10.0 cm?

77. A spring is standing upright on a table with its bottom end fastened to the table. A block is dropped from a height 3.0 cm above the top of the spring. The block sticks to the top end of the spring and then oscillates with an amplitude of 10 cm. What is the oscillation frequency?

78. The analysis of a simple pendulum assumed that the mass was a particle, with no size. A realistic pendulum is a small, uniform sphere of mass M and radius R at the end of a massless string, with L being the distance from the pivot to the center of the sphere.
    a. Find an expression for the period of this pendulum.
    b. Suppose $M = 25$ g, $R = 1.0$ cm, and $L = 1.0$ m, typical values for a real pendulum. What is the ratio T_{real}/T_{simple}, where T_{real} is your expression from part a and T_{simple} is the expression derived in this chapter?

79. a. A mass m oscillating on a spring has period T. Suppose the mass changes very slightly from m to $m + \Delta m$, where $\Delta m \ll m$. Find an expression for ΔT, the small change in the period. Your expression should involve T, m, and Δm but not the spring constant.
    b. Suppose the period is 2.000 s and the mass increases by 0.1%. What is the new period?

80. FIGURE CP14.80 shows a 200 g uniform rod pivoted at one end. The other end is attached to a horizontal spring. The spring is neither stretched nor compressed when the rod hangs straight down. What is the rod's oscillation period? You can assume that the rod's angle from vertical is always small.

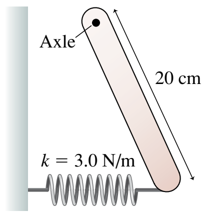


FIGURE CP14.80

STOP TO THINK ANSWERS

Stop to Think 14.1: c. $v_{max} = 2\pi A/T$. Doubling A and T leaves v_{max} unchanged.

Stop to Think 14.2: d. Think of circular motion. At 45°, the particle is in the first quadrant (positive x) and moving to the left (negative v_x).

Stop to Think 14.3: c > b > a = d. Energy conservation $\frac{1}{2}kA^2 = \frac{1}{2}m(v_{max})^2$ gives $v_{max} = \sqrt{k/m}\,A$. k or m has to be increased or decreased by a factor of 4 to have the same effect as increasing or decreasing A by a factor of 2.

Stop to Think 14.4: c. $v_x = 0$ because the slope of the position graph is zero. The negative value of x shows that the particle is left of the equilibrium position, so the restoring force is to the right.

Stop to Think 14.5: c. The period of a pendulum does not depend on its mass.

Stop to Think 14.6: $\tau_d > \tau_b = \tau_c > \tau_a$. The time constant is the time to decay to 37% of the initial height. The time constant is independent of the initial height.